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BARGAINING GAME THEORY

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BARGAINING GAME THEORY

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CONTENTS

cl : closure; com : comprehensive hull; comv : comprehensive convex hull; conv : convex hull; int : interior; relint : relative interior

$\vee$: maximum; $\wedge$: minimum; $|A|$: number of elements in A; $A \setminus B = \{x \in A : x \notin B\}$ for arbitrary sets A and B.

For $x, y \in \mathbb{R}^n$: $x > y$ means $x_i > y_i$ for every $i = 1,2,\ldots,n$
$x \geqq y$ means $x_i \geqq y_i$ for every $i = 1,2,\ldots,n$
$x < y$ means $x_i < y_i$ for every $i = 1,2,\ldots,n$
$x \leqq y$ means $x_i \leqq y_i$ for every $i = 1,2,\ldots,n$

For a function $f : T \subset \mathbb{R} \to \mathbb{R}$: "increasing" means : $x > y \Rightarrow f(x) > f(y)$
"nondecreasing" means : $x > y \Rightarrow f(x) \geqq f(y)$
"decreasing" means : $x > y \Rightarrow f(x) < f(y)$
"nonincreasing" means : $x > y \Rightarrow f(x) \leqq f(y)$

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geqq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$, where $0 = (0,0,\ldots,0) \in \mathbb{R}^n$.
Similarly for $\mathbb{Q}^n_+$, $\mathbb{Q}^n_{++}$ ($\mathbb{Q}$: rational numbers)

$L(A)$: set of lotteries $[p_1;a^1,p_2;a^2,\ldots,p_n;a^n]$ over A
$U(A)$: family of functions $u : A \to \mathbb{R}$
Eu : expected utility under u
$N = \{1,2,\ldots,n\}$ set of players
vNM : von Neumann - Morgenstern

$\pi^i : \mathbb{R}^n \to \mathbb{R}$ projection on i-th coordinate
$\pi_{-i} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ omit i-th coordinate

f^i_S : Pareto function

$\pi x = (x_{\pi(1)}, x_{\pi(2)},\ldots,x_{\pi(n)})$ for a permutation π of N and $x \in \mathbb{R}^n$

$x.y$ inner product of $x, y \in \mathbb{R}^n$
$xy = (x_1y_1,x_2y_2,\ldots,x_ny_n)$ for $x, y \in \mathbb{R}^n$

$xT = \{xy : y \in T\}$ for $T \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$

$S + T = \{s + t : s \in S,\ t \in T\}$ for $S, T \subset \mathbb{R}^n$

$E^N(z)$, $E^N(T)$: see p.128,129.

$BS = BS^2$: family of 2-person bargaining situations

BS^n : family of n-person bargaining situations

$BSC = BSC^2$: family of 2-person bargaining situations with only riskless Pareto optimal outcomes

BSC^n : family of n-person bargaining situations with only riskless Pareto optimal outcomes

B_+SC : subfamily of BSC with corresponding bargaining games in B_+

$B = B^2$: family of 2-person bargaining games

B^n : family of n-person bargaining games

$B_+^n, B_+^2 = B_+$: bargaining games S with $S = S_+$

$S_+ = \text{com}(S \cap \mathbb{R}_+^n)$ $(S \in B^n)$

S_Γ : bargaining game corresponding to $\Gamma \in BS^n$

$\Delta = \Delta^2 = \text{comv}\{(1,0),(0,1)\}$

$\Delta^n = \text{comv}\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\} \subset \mathbb{R}^n$

$\Box = \Box^2 = \text{com}\{(1,1)\}$

$\Box^n = \text{com}\{(1,1,\ldots,1)\} \subset \mathbb{R}^n$

$g(S)$: utopia point of S $(S \in B^n)$

$h(S)$: global utopia point of S $(S \in B^n)$

$\text{alt}(\phi,\Gamma)$: set of ϕ-alternatives of Γ $(\Gamma \in BS^n)$

$C^1(\Gamma)$, $K^1(\Gamma)$: see p.102 $(\Gamma \in BS^n)$

$C^1(S)$, $K^1(S)$: see p.110 $(S \in B^n)$

$N^t, N^{\frac{1}{2}} = N$: Nash solutions

D : disagreement solution

$D^1 = \underline{p}$, $D^2 = \bar{p}$: dictator solutions

KS : Kalai - Smorodinsky solution

KR : Kalai - Rosenthal solution

E^p : egalitarian solution

$\bar{w}(S)$: point of S with first coordinate 0 and maximal second coordinate ($S \in B$)

$\underline{w}(S)$: point of S with second coordinate 0 and maximal first coordinate ($S \in B$)

$\bar{W}(S) = \mathrm{conv}\{\bar{w}(S), \bar{p}(S)\}$, $\underline{W}(S) = \mathrm{conv}\{\underline{w}(S), \underline{p}(S)\}$ ($S \in B$)

P(S) : Pareto optimal subset of $S \in B^n$

$W(S) = \partial S$ ($S \in B^n$)

N : family of Nash solutions, $\bar{N}$: $N \cup \{\underline{w}, \bar{w}, D\}$

AN : Anonymity

CC : Convention consistency

CIIA : Conditional independence of irrelevant alternatives

CONS : Consistency

GIM : Global individual monotonicity

HOM : Homogeneity

IIA : Independence of irrelevant alternatives

IIE : Independence of irrelevant expansions

IM : Individual monotonicity

INIR : Independence of non-individually rational outcomes

IR : Individual rationality

PCO : Pareto continuity

PO : Pareto optimality

PSA : Partial super-additivity

RA : Restricted additivity

RM : Restricted monotonicity

RPO : Risk profit opportunity

RS : Risk sensitivity

SA : Super-additivity

SIR : Strong individual rationality

SL : Slice property

STI : Scale transformation invariance

SYM : Symmetry

TS : Twist sensitivity

WA : Worse alternative property

WPO : Weak Pareto optimality

■ denotes the end of a proof.

PREFACE

This monograph is about bargaining game theory, more precisely : about - what are often called - *axiomatic* models of bargaining. In these models the approach initiated by Nash in his 1950 paper, is followed. Roughly : *Solutions* for *bargaining games* are characterized by their properties, relations between such properties are studied. This bargaining game theory is a branch of game theory, more specifically, it belongs to the theory of cooperative games without side-payments. Game theory is a mathematical discipline which concerns itself with the study of mathematical models of situations where conflict is involved. It has applications in many fields, especially in economics and social sciences.

An earlier monograph on the subject of the present work is Roth (1979). Survey papers on the subject are Schmitz (1977), Kalai (1983), and Peters (1983).

General works on game theory are : (the seminal) von Neumann and Morgenstern (1944), Luce and Raiffa (1957), Vorobo'ev (1977), Rosenmüller (1981), Owen (1982), Shubik (1982).

The reader will find a summary and the plan of this monograph in section 3 *), chapter 1, where these may fit in best, since, then, section 1 has already presented a first acquaintance with the bargaining problem. Readers who are in a hurry, may skip the first two chapters (Introduction and Utility Theory). Those who are not familiar with the von Neumann - Morgenstern utility theory, however, should read section 4 in chapter 2.

*) Sections are numbered consecutively throughout the monograph. E.g., the second chapter starts with section 4.

CHAPTER 1

INTRODUCTION

This introductory chapter consists of three sections. In section 1, we present, in a more or less informal way, some examples of situations which may give rise to the existence of bargaining games. Section 2 contains some historical remarks, and section 3 presents a summary and the plan of this monograph.

1. EXAMPLES OF SITUATIONS WHICH MAY LEAD TO BARGAINING GAMES

We shall give a few examples of situations which may give rise to the existence of bargaining games. We use the word *situation* here, informally, in its everyday meaning (; in the next chapter a - formal - definition of a bargaining situation will be given). Throughout this section, we shall maintain a more or less informal level of presentation : the ideas rather than their formalizations, are central here.

For a story behind the first example, see, e.g. Luce and Raiffa (1957).

Example 1.1. (Prisoner's Dilemma). Consider the *bimatrix game* given by the following diagram :

$$\begin{array}{cc} & \text{Player 2} \\ & \begin{array}{cc} \quad L & \qquad R \end{array} \\ \text{Player 1} \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} (5,5) & (0,6) \\ (6,0) & (1,1) \end{bmatrix} \end{array}$$

(In any of the four pairs, the first and second numbers are the payoffs to players 1 and 2, respectively.) Both player 1 and player 2 have two *pure strategies* (top (T) and bottom (B) row, and left (L) and right (R) column, respectively), and an infinity of *mixed strategies*, e.g. for player 1 : play T with probability p and B with probability $1-p$ (where $p \in [0,1]$). As an example, the payoffs to the players, if player 1 plays T with probability $\frac{1}{3}$ and B with probability $\frac{2}{3}$ and player 2 plays L with probability $\frac{1}{2}$ and R with probability $\frac{1}{2}$, are the (expected) payoffs

$\frac{1}{3}(\frac{1}{2}\cdot 5+\frac{1}{2}\cdot 0)+\frac{2}{3}(\frac{1}{2}\cdot 6+\frac{1}{2}\cdot 1)=3\frac{1}{6}$ for player 1 and

$\frac{1}{2}(\frac{1}{3}\cdot 5+\frac{2}{3}\cdot 0)+\frac{1}{2}(\frac{1}{3}\cdot 6+\frac{2}{3}\cdot 1)=2\frac{1}{6}$ for player 2. The unique *Nash equilibrium* (Nash (1951)) of this game is the pair of pure strategies (B,R) : no player gains by deviating, unilaterally, from this pair of strategies. Both players, however, strictly prefer the pair of payoffs (5,5) corresponding to the pair of strategies (T,L), to the payoff pair (1,1) corresponding to the Nash equilibrium of the game. The pair of payoffs (5,5) is unlikely to be the final outcome of the game unless the players have a way to make some binding agreement (e.g., sign a contract) to play (T,L).

By using mixed strategies, the players can achieve any pair of payoffs in the shaded area S in Fig. 1.1.

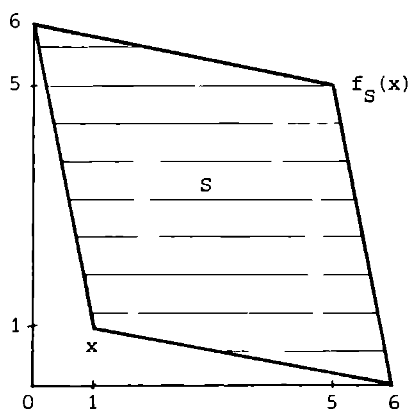


Figure 1.1.

Let $f_S : S \to S$ be a map, and suppose that the players agree to obey the following procedure : every player announces a mixed strategy, and the corresponding payoff pair $x \in S$ is calculated; and then a so-called *correlated strategy* is determined and carried out such that the corresponding pair of payoffs is

$f_S(x)$. Such a correlated strategy has the form $(z_{TL}, z_{TR}, z_{BL}, z_{BR})$ where, e.g., z_{TL} is the joint probability that player 1 has to play T and player 2 has to play L. In this particular example, S is also the set of payoffs corresponding to correlated strategies. Suppose that f_S assigns to $x \in S$ the (by both players) most preferred point in S on the 45°-line through x. If, e.g., the players play the equilibrium pair of strategies (B,R), then $x = (1,1)$, $f_S(x) = (5,5)$, and $f_S(x)$ can be achieved by the correlated strategy (1,0,0,0). If we have, for every set S of this kind, a prespecified map f_S as above and a point x_S in S, then we can define a map ϕ which assigns to every S the point $f_S(x_S)$. The purpose of this monograph is to study properties of such maps ϕ.

Example 1.2. (Battle of the sexes; see, e.g., Luce and Raiffa (1957)). Consider the following bimatrix game :

$$\begin{array}{ccc} & & \text{Player 2} \\ & & \begin{array}{cc} \text{L} & \quad\text{R} \end{array} \\ \text{Player 1} & \begin{array}{c} \text{T} \\ \text{B} \end{array} & \begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{bmatrix} \end{array}$$

The shaded area in Fig. 1.2 is the set of payoffs attainable by mixed strategies. The set of payoffs attainable by correlated strategies is $\text{conv}\{(0,0),(2.1),(1,2)\} =: S$. (We denote by "conv" the convex hull.)

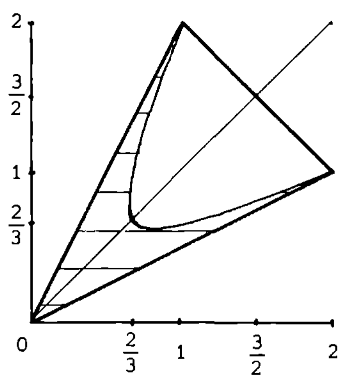


Figure 1.2.

For this game, there are three Nash equilibria : (T,L), (B,R), and (M,N), where M means : play T with probability $\frac{2}{3}$, and N means : play R with probability $\frac{2}{3}$. The corresponding payoff pairs are (2,1), (1,2), and $(\frac{2}{3},\frac{2}{3})$, respectively, and if f_S is as in Eample 1.1, then $f_S(2,1) = (2,1)$, $f_S(1,2) = (1,2)$, and $f_S(\frac{2}{3},\frac{2}{3}) = (\frac{3}{2},\frac{3}{2})$; these payoffs can be achieved by the correlated strategies (1,0,0,0), (0,0,0,1), and $(\frac{1}{2},0,0,\frac{1}{2})$, respectively. The payoff pair $(\frac{3}{2},\frac{3}{2})$ can only be obtained by a correlated strategy and not by a pair of mixed strategies; consequently, it can only be obtained by cooperation between the players since both of them have to consent to the use of the correlated strategy $(\frac{1}{2},0,0,\frac{1}{2})$.

A bimatrix game is a *noncooperative game*, and the Nash equilibrium concept a *noncooperative solution concept*; a map ϕ as introduced at the end of Example 1.1, is a *cooperative solution concept*, and we shall formally introduce it as a *bargaining solution* in the third chapter. Whether we call a game cooperative or not, depends on the rules of the game. An essential feature of cooperation is the possibility of binding agreements.

Note that the game procedure, as a whole, of Examples 1.1 and 1.2, is of a noncooperative nature : once the players have agreed to the use of correlated strategies and a "bargaining solution" ϕ, the final payoffs depend only on the strategies announced by the players. In literature, these games are called *arbitration games*, and they were introduced by Nash (1953). See also Raiffa (1953), Luce and Raiffa (1957), and for a comprehensive study, see Tijs and Jansen (1980, 1982), and Jansen (1981).

If we use the word *alternative* instead of *pair of pure strategies* (in some bimatrix game), *lottery* instead of *correlated strategy*, *disagreement alternative* for some fixed pair of pure or mixed strategies, and *utilities* instead of *payoffs*, then we have all the ingredients of a *bargaining situation*. In the major part of this monograph we do not consider bargaining situations, but *bargaining games* : i.e. we consider utilities instead of alternatives. All these concepts will be formally introduced in the next chapters.

We conclude this section with an example of a different nature than the previous examples.

Example 1.3. Consider the situation where one dollar has to be divided between two players 1 and 2. We suppose that the *utility* of receiving x dollars is x for player 1 and $\sqrt{x}$ for player 2. (The concept of *utility* will be formally defined in the next chapter.) If the players reach an agreement on the division of the dollar, say x for player 1 and y for player 2, with $x + y \leqq 1$, then player 1 receives utility x and player 2 receives $\sqrt{y}$. Otherwise, the game ends in disagreement, leaving each player with 0 dollars or *utility payoff* 0. Fig. 1.3 depicts the set S of possible utility payoffs for this game.

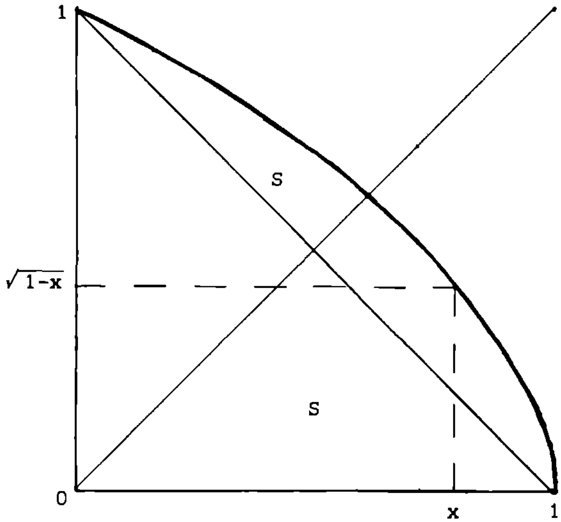


Figure 1.3.

Here the utility pair $(\frac{1}{2},\frac{1}{2})$ may be obtained by a money allocation $(\frac{1}{2},\frac{1}{4})$, leaving one quarter of the dollar unallocated, but also by a lottery in which each player has a 50 percent chance of getting the dollar.

By not cooperating, each player can secure himself of (at most) utility 0. The map f_S as in Example 1.1 would assign the point $(\frac{1}{2}\sqrt{5} - \frac{1}{2}, \frac{1}{2}\sqrt{5} - \frac{1}{2})$ to (0,0) : we will later (section 15) recognize that point as the "Kalai - Smorodinsky - solution of the bargaining game".

2. SOME HISTORICAL REMARKS

Von Neumann and Morgenstern (1944) have proposed a "solution" for cooperative games with sidepayments : a similar solution for bargaining games would assign to a game the set of all Pareto optimal individually rational outcomes, for instance, to the game of Example 1.3, the set $\{(x,\sqrt{1-x}) : 0 \leqq x \leqq 1\}$. (Formal definitions follow in chapter 3.) A drawback of this "solution" is its lack of uniqueness.

Nash (1950) has proposed a solution concept which assigns exactly one point to each bargaining game. We will for the first time encounter that solution in section 11. Other stepping stones in the development of (axiomatic) bargaining game theory are : Raiffa (1953), Kalai and Smorodinsky (1975), Kalai (1977), Perles and Maschler (1981). In all these papers, new solution concepts were proposed or characterizations given, which have become more or less standard. Surveys are Schmitz (1977), Roth (1979), Peters (1983). Another survey-like paper is Thomson and Myerson (1980).

So far, we have used the word *axiomatic* a few times, in conjunction with *bargaining game theory*. However, we prefer to use the word *property* instead of *axiom* in this monograph. Nevertheless, for historical reasons, we adopt the expression *axiomatic bargaining game theory*.

There are other approaches to bargaining than the one offered by axiomatic bargaining game theory. For instance, there are noncooperative approaches: we mention Nash (1953), and, as a more recent contribution, Rubinstein (1982). A recent work containing many different approaches, was edited by Roth (1985). Of course, there are many works on bargaining which are less formal than game-theoretical works; we mention Schelling (1960) and Raiffa (1982).

3. SUMMARY AND PLAN OF THIS MONOGRAPH

Chapter 2 discusses the elements of utility theory, needed in this monograph : the von Neumann - Morgenstern utility theorem in section 4, risk aversion in section 5, additive and multiplicative utility in sections 6 and 7, respectively. Section 4 contains groundwork for the whole monograph, section 5 for chapter 8 and section 30, which both deal with risk properties of bargaining solutions; and the results of sections 6 and 7 are used in chapter 6 (on additivity properties) and section 12 (on characterizations of so-called Nash solutions), respectively.

Chapter 3 introduces the central concepts of this monograph : bargaining situations in section 8, bargaining games in section 9, and bargaining solutions in section 10.

Chapter 4 deals almost exclusively with the (2-person) Nash bargaining solution and its nonsymmetric extensions. In section 11, the independence of irrelevant alternatives property is central, in section 12 two properties called independence of irrelevant expansions, and convention consistency are considered. All mentioned properties are used in characterizations. Section 13 proposes a simple market model and gives relations between the (noncooperative) solution concept for this model, competitive equilibrium, and bargaining solutions, in particular Nash solutions. Section 14 studies another (noncooperative) model which gives rise to nonsymmetric Nash solutions; the presentation there is not completely rigorous.

Chapter 5 gives a characterization of a family of nonsymmetric extensions of the (2-person) Kalai - Smorodinsky solution in section 15, and of the (2-person) Kalai - Rosenthal solution in section 16. Individual monotonicity is the central property in section 15, global individual monotonicity in section 16.

Chapter 6 deals with additivity properties of 2-person bargaining solutions. Section 17 provides a utility-theoretic foundation for the use of such properties, with the aid of the result of section 6. In section 18, a characterization of the family of so-called proportional solutions is given which uses the (partial) super-additivity property; section 20 provides another characterization of the family of nonsymmetric Nash solutions of chapter 4, with the aid of the restricted additivity property. Section 19 is an intermezzo on (Pareto-) continuity of bargaining solutions.

Chapter 7 considers two other approaches to bargaining : multisolutions ("multivalued bargaining solutions"), and probabilistic solutions. The latter assign a probability measure, instead of one fixed outcome, to every bargaining game. In section 21, (2-person) multisolutions with an independence of irrelevant alternatives property are considered and an extension is obtained of the main result of section 11, and also multisolutions with a restricted monotonicity property are studied, which gives an extension of the characterization in section 15. Section 22 discusses two versions of an independence of irrelevant alternatives property for probabilistic solutions; again, an extension is obtained of the characterization in section 11.

Chapter 8 studies risk properties and relations between risk properties and other properties for 2-person bargaining solutions. The groundwork for the definitions of these risk properties was done in section 5. Section 23 mainly discusses the risk sensitivity property and the worse alternative property, and section 24 the relations between these properties and other properties like independence of irrelevant alternatives and individual monotonicity. In section 25, the twist sensitivity and slice properties are introduced, and a relation with risk sensitivity established. Most results until section 26 hold for games where every Pareto optimal outcome is riskless : in section 26, the case where a Pareto optimal outcome may be attainable only by a lottery, is considered. Until section 27, the main question of chapter 8 is : which are the effects on the solution outcome if a player in a bargaining game is replaced by a more risk averse player ? Section 27 briefly touches the strategical question whether it may be advantageous for a player to pretend to be more or less risk averse, if that is allowed by the rules of the game. The so-called b-monotonicity property, introduced in section 13, plays a central role here.

Chapter 9 extends some of the results of chapters 4, 5, and 8, to the n-person case. In section 28, the independence of irrelevant alternatives property is considered, in section 29 the individual monotonicity property, and in section 30, risk properties for n-person bargaining solutions are studied. The concluding chapter 10 contains only one section (31), which gives, in some diagrams, an overview of the main relations between properties of bargaining solutions, as established in this monograph.

UTILITY THEORY

We review and modify some results in utility theory needed elsewhere in this monograph. In section 4, we consider von Neumann - Morgenstern utility functions, in section 5 a measure of risk aversion which we need when we study risk properties of bargaining solutions. The result of section 6 on additive utility will be applied in chapter 6, especially section 17; and the result of section 7 on multiplicative utility will be used in section 12. One may postpone reading sections 5 - 7 until reference is made to these sections.

4. VON NEUMANN - MORGENSTERN UTILITY FUNCTIONS

Let L be an arbitrary nonempty set (of decisions, strategies, alternatives, prospects, commodity bundles, ...). A *preference relation* $\succcurlyeq$ on L is a binary relation on L which is complete, i.e. $m \succcurlyeq k$ or $k \succcurlyeq m$ for all $k,m \in L$, and transitive. We denote $m \succ k$ if not $k \succcurlyeq m$, and $k \approx m$ if $k \succcurlyeq m$ and $m \succcurlyeq k$, for all $k,m \in L$. For $m,k \in L$, we pronounce $m \succcurlyeq k$ as : k is not preferred to m; $m \succ k$ as : m is preferred to k; $m \approx k$ as : m and k are equivalent. (Indeed $\approx$ is an equivalence relation.)

The situation prevailing in this monograph is that $L = L(A)$ is the set of probability measures with finite support on a nonempty set A. A typical element of L(A) is denoted

$$\ell = [p_1;a^1,p_2;a^2,\ldots,p_n;a^n] = [p_i;a^i]_{i=1}^n$$

where $n \in \mathbb{N}$, $a^i \in A$ and $p_i \geq 0$ for every $i = 1,2,\ldots,n$, $\Sigma_{i=1}^n p_i = 1$. Of course, ℓ is the *lottery* which results with probability $\Sigma_{j:a^j = a^i} p_j$ in a^i, for every $i = 1,2,\ldots,n$. With ℓ as above, we shall also use the notation $[\ldots,q;\ell,\ldots]$ for $[\ldots,qp_1;a^1,qp_2;a^2,\ldots,qp_n;a^n,\ldots]$. Further, by writing $[1;a]$ for $a \in A$, we have $A \subset L(A)$.

We call L(A) the *lottery set of* A. Elements of A are called *riskless alternatives*, elements of $L(A)\setminus A$ *risky alternatives*.

Let U(A) denote the family of all functions $u : A \to \mathbb{R}$. For $\ell = [p_i;a^i]_{i=1}^n \in L(A)$ and $u \in U(A)$, we denote by $Eu(\ell) := \Sigma_{i=1}^n p_i u(a^i)$ the *expected utility of* ℓ *(under* u*)*. Let $\succcurlyeq$ be a preference relation on L(A).

Under certain conditions on $\succ$, it can be proved (see, e.g., Herstein and Milnor (1953)) that there exists a function u in U(A) such that

(4.1) $\ell \succ m \Leftrightarrow Eu(\ell) \geq Eu(m)$ for all $\ell, m \in L(A)$

and moreover

(4.2) if (4.1) is also satisfied with $v \in U(A)$ in the role of u, then there are $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $v(a) = \alpha u(a) + \beta$ for all $a \in A$.

A function $u \in U(A)$ satisfying (4.1) and (4.2) is called a *von Neumann - Morgenstern* (vNM) *utility function (for* $\succ$, or : *representing* $\succ$*)*. In this monograph, utility functions will, in most cases, be assumed to be of the vNM-type.

We conclude this section with a simple example which links the present section to the following one. Let $A = [0,1] \subset \mathbb{R}$, and let $u \in U(A)$ be defined by $u : x \mapsto \sqrt{x}$. Then

$u(\frac{1}{2}) = \frac{1}{2}\sqrt{2} > \frac{1}{2} = Eu([\frac{1}{2};0,\frac{1}{2};1])$,

so an individual with this vNM-utility function u prefers obtaining $\frac{1}{2}$ (e.g., dollar) with certainty to a lottery in which he has a 50 % chance of obtaining 0 and a 50 % chance of obtaining 1. This kind of preference is exhibited by a *risk averse* individual. In the following section we will discuss a related concept : the relation on U(A) called *more risk averse than.*

5. RISK AVERSION

Suppose, an individual may choose between receiving \$ 5 for certain, and a lottery ticket which gives him \$ 10 or nothing both with a fifty percent chance. An individual that is called risk averse (-neutral, -loving, respectively) in literature, will prefer the dollars (be indifferent, prefer the lottery ticket, respectively).

Here, we are not so much interested in some absolute measure of risk aversion of a decision maker, but rather we look for a way to compare the aversion to risk of two decision makers. We shall introduce a relation *more risk averse*, and derive a mathematical characterization of this relation.

Pioneering work in the area of risk aversion was done by Arrow (see Arrow (1971)) and Pratt (1964). Other important contributions are Yaari (1969) and Kihlstrom and Mirman (1974).

We shall follow Yaari's approach with a minor - but for our purposes important - modification. Further, our characterization theorem (characterizing the relation more risk averse) is more general than the characterizations usually found in the literature, since it does not need to assume any continuity or differentiability properties of the utility functions, and (hence) no topological or algebraic structure on the set of alternatives.

In this section, A will always be a nonempty set of alternatives, with corresponding lottery set L(A).

<u>Definition 5.1.</u> (i) For $u \in U(A)$ and $a \in A$, we call the set

$P_u(a) := \{\ell \in L(A) : Eu(\ell) > u(a)\}$

the *preference set of* u *with respect to* a.

(ii) Let $u,v \in U(A)$. We call (the decision maker with utility function) v *more risk averse than* (the decision maker with utility function) u (notation : vMRu) if $P_v(a) \subset P_u(a)$ for every $a \in A$.

As to (i) of Def. 5.1, we note that for $\alpha,\beta \in \mathbb{R}$ with $\alpha > 0$, we have $P_{\alpha u+\beta}(a) = P_u(a)$ for every $a \in A$, so preference sets are independent of the particular representation chosen (cf. (4.2)). Hence also the relation MR above is independent of the chosen representations u and v. Def. 5.1 (ii) is a small modification of the definition proposed by Yaari (1969), who uses so-called acceptance sets $A_u(a) := \{\ell \in L(A) : Eu(\ell) \geq u(a)\}$ instead of preference sets $P_u(a)$. The reason for this modification is, that (iii) of the following lemma would not hold with acceptance sets instead of preference sets in the definition of MR.

<u>Lemma 5.2.</u> Let $u,v \in U(A)$ with vMRu, and $a,b \in A$. Then :

(i) if $v(b) > v(a)$, then $u(b) > u(a)$,

(ii) if $u(a) \leq u(b)$, then $v(a) \leq v(b)$,

(iii) if $u(a) = u(b)$, then $v(a) = v(b)$.

<u>Proof</u>. Suppose $v(b) > v(a)$. Then $b \in P_v(a) \subset P_u(a)$. So $u(b) > u(a)$. (ii) follows immediately from (i). For (iii), we note that (i) implies : if $v(a) \neq v(b)$, then $u(a) \neq u(b)$.■

The following definition extends the concept of concavity of a function.

We need that extension in the characterization theorem of MR.

Definition 5.3. Let T be a nonempty subset of $\mathbb{R}^{\ell}$ ($\ell \in \mathbb{N}$). We call a function $k : T \to \mathbb{R}$ *concave* if

$k(\Sigma_{i=1}^{m} p_i t^i) \geq \Sigma_{i=1}^{m} p_i k(t^i)$

for any convex combination $\Sigma_{i=1}^{m} p_i t^i$ of elements of T which is itself an element of T.

Lemma 5.4. Let $T \subset \mathbb{R}$, and $k : T \to \mathbb{R}$ nondecreasing and concave. Let $\Sigma_{i=1}^{m} p_i t^i$ be a convex combination of elements of T, not necessarily in T, and let $t \in T$ be such that $\Sigma_{i=1}^{m} p_i t^i \leq t$. Then we have $\Sigma_{i=1}^{m} p_i k(t^i) \leq k(t)$.

Proof. If $t^i \leq t$ for every i, then the proof is finished by nondecreasingness of k. Otherwise, w.l.o.g. $t^1 > t$. Take $0 \leq \lambda < 1$ such that

$\lambda t^1 + (1-\lambda)\Sigma_{i=1}^{m} p_i t^i = t$. By concavity of k, we obtain

$k(t) \geq \lambda k(t^1) + (1-\lambda)\Sigma_{i=1}^{m} p_i k(t^i)$. So

$\Sigma_{i=1}^{m} p_i k(t^i) \leq (k(t)-\lambda k(t^1))(1-\lambda)^{-1}$,

which implies $\Sigma_{i=1}^{m} p_i k(t^i) \leq k(t)$ because, by nondecreasingness of k, $k(t^1) \geq k(t)$.■

If vMRu for v and u in U(A), then, in view of Lemma 5.2 (iii), we can define the function $k : u(A) \to \mathbb{R}$ by $k(u(a)) = v(a)$ for every $a \in A$. This fact will be used in the following theorem, which is the announced characterization theorem.

Theorem 5.5. Let $u,v \in U(A)$. The following two assertions are equivalent :

(i) vMRu

(ii) The function $k : u(A) \to \mathbb{R}$ with $k(u(a)) = v(a)$ for every $a \in A$, is well-defined, nondecreasing, and concave.

Proof. (a) Suppose that (i) holds. Then the function k in (ii) is well-defined and nondecreasing by (iii) and (ii), respectively, of Lemma 5.2. It remains to be shown that k is concave. Let u(a) be the convex combination $\Sigma_{i=1}^{m} p_i u(a^i)$ of $u(a^1), u(a^2), \ldots, u(a^m) \in u(A)$. We have to prove that

$k(u(a)) \geq \Sigma_{i=1}^{m} p_i k(u(a^i))$

or that

$v(a) \geq \Sigma_{i=1}^{m} p_i v(a^i) = Ev([p_i;a^i]_{i=1}^{m})$.

If $v(a) < Ev([p_i;a^i]_{i=1}^m)$, then $[p_i;a^i]_{i=1}^m \in P_v(a) \subset P_u(a)$, which implies the contradiction

$$u(a) < Eu([p_i;a^i]_{i=1}^m) = \Sigma_{i=1}^m p_i u(a^i) = u(a).$$

So k is concave, and (ii) holds.

(b) Suppose that (ii) holds. We want to show that $P_v(a) \subset P_u(a)$ for every $a \in A$. Let $a \in A$ and $\ell = [p_i;a^i]_{i=1}^m \in L(A)$. If we have $Eu(\ell) \leq u(a)$, then, by Lemma 5.4, we also have $Ev(\ell) = \Sigma_{i=1}^m p_i v(a^i) = \Sigma_{i=1}^m p_i k(u(a^i)) \leq k(u(a)) = v(a)$. In other words, $\ell \notin P_u(a) \Rightarrow \ell \notin P_v(a)$, or $P_v(a) \subset P_u(a)$. ∎

In Peters and Tijs (1981), Theorem 5.5 (see Theorem 3.2 there) was proved by using, in part (b) of the proof, an extension of the function k to conv(u(A)). This extension could easily be shown to exist since the set A was assumed to be a compact subset of $\mathbb{R}^\ell$ and attention was restricted to continuous elements of U(A).

Theorem 5.5 states that if a utility function v is more risk averse than a utility function u, then $v = k \circ u$ where k is a nondecreasing concave function on u(A). It will sometimes simplify presentation, in the sequel, if we have k defined on conv(u(A)) (= $\{Eu(\ell) : \ell \in L(A)\}$). We shall prove that the function $k : u(A) \to \mathbb{R}$ can be extended to a function $\tilde{k} : \text{conv}(u(A)) \to \mathbb{R}$, also without requiring compactness of A or continuity of u. This result will follow as a corollary of Lemma 5.6 below. (Actually, we do not need that lemma in such a general form, since, in the sequel, we will nevertheless assume compactness of $A \subset \mathbb{R}^\ell$ and continuity of u.)

<u>Lemma 5.6.</u> Let $T \subset \mathbb{R}$, $T \neq \emptyset$, and let $k : T \to \mathbb{R}$ be a nondecreasing, concave, function. Then there exists a nondecreasing concave function $\tilde{k} : \text{conv}(T) \to \mathbb{R}$ such that $\tilde{k}(t) = k(t)$ for all $t \in T$.

<u>Proof</u>. Let M be the family of functions $f : \text{conv}(T) \to \mathbb{R}$ that are nondecreasing, concave, and have $f(t) \geq k(t)$ for every $t \in T$. For every $t \in T$ except the smallest t (if that exists) we will construct a $g_t \in M$ with $g_t(t) = k(t)$. Then we take for the function $\tilde{k}$ the pointwise infimum of the functions in M, except for the smallest element $\underline{t}$ (if it exists) where we define $\tilde{k}(\underline{t}) := k(\underline{t})$. This function $\tilde{k}$ has the required properties, as is elementarily verified. We still have to construct the functions g_t.

If T has a maximum $\bar{t}$, then $g_{\bar{t}}(x) := k(\bar{t})$ for every $x \in \text{conv}(T)$. Let now $t \in T$, such that there are s, s' in T with $s < t < s'$. Let $\hat{f}$ be the affine function with $\hat{f}(s) = k(s)$ and $\hat{f}(t) = k(t)$. The concavity of k implies that $\hat{f}(p) \geq k(p)$ for every p in T with $p \geq t$. Then let g_t be the affine function with $g_t(t) = k(t)$ and with slope equal to the infimum of the slopes of all affine functions f which have $f(t) = k(t)$ and $f(p) \geq k(p)$ for every p in T with $p \geq t$. (Such a function f exists, namely $f = \hat{f}$, and also the infimum exists because 0 is lower bound for all those slopes, in view of the nondecreasingness of k.) Then by definition we have $g_t(p) \geq k(p)$ for every p in T with $p \geq t$.Suppose there is a $\underline{p} < t$ with $g_t(\underline{p}) < k(\underline{p})$. Let $\underline{f}$ be the affine function with $\underline{f}(\underline{p}) = k(\underline{p})$ and $\underline{f}(t) = k(t)$. Then the slope of f is smaller than the slope of g_t whereas, by the concavity of k, $\underline{f}(p) \geq k(p)$ for all p in T with $p \geq t$. From this contradiction we conclude that $g_t(p) \geq k(p)$ also for all p in T with $p < t$, so $g_t \in M$. ■

<u>Corollary 5.7.</u> For $u \in U(A)$ and $k : u(A) \to \mathbb{R}$ nondecreasing and concave, there exists a $\tilde{k} : \mathrm{conv}(u(A)) \to \mathbb{R}$ nondecreasing and concave, such that $\tilde{k}(u(a)) = k(u(a))$ for every $a \in A$.

6. ADDITIVE UTILITY

Keeney and Raiffa (1976, p. 231), following Fishburn (1965), give a necessary and sufficient condition under which a von Neumann - Morgenstern utility function on the Cartesian product of two sets of alternatives, can be written as a scaled sum of coordinate utility functions. In this section, we shall modify this result for the case where these coordinate utility functions represent given preference relations. The motivation for including this result is, that it provides a utility-theoretic foundation for additivity properties of bargaining solutions, in chapter 6. The result was published as Peters (1985).

In order to specify the problem we deal with in this section, let A and B be non-empty sets of alternatives, and let $C := A \times B$ be the Cartesian product of these sets. $L(A)$, $L(B)$, and $L(C)$ denote the corresponding lottery sets. Let $\succ_C$ be a preference relation on $L(C)$ representable by a vNM utility function $w : C \to \mathbb{R}$. Keeney and Raiffa (1976, p. 231) show that under the assumption of *additive independence* for $\succ_C$ (see below) we can write $w = k_A w_A + k_B w_B$ where k_A and k_B are positive constants and w_A and w_B are induced vNM utility functions on A and B. We shall modify this result for the case where w_A and w_B represent given preference relations on L(A) and L(B).

Let now A, B and C be as above, and let $\succ_A, \succ_B$, and $\succ_C$ be preference relations on L(A), L(B), and L(C), respectively. We assume :

(6.1) Any preference relation occuring in this section is representable by a vNM utility function.

We introduce a weaker version of the *additive independence* property (cf. Keeney and Raiffa (1976, p. 230)). Our version is weaker since we consider only lotteries with probabilities $\frac{1}{2}$.

(6.2) For all (a,b) and (a',b') in C, we have :

$$[\tfrac{1}{2};(a,b),\tfrac{1}{2};(a',b')] \approx_C [\tfrac{1}{2};(a,b'),\tfrac{1}{2}(a',b)].$$

(*Additive independence*)

We shall show that a decision maker with an additively independent $\succ_C$ is indifferent between a lottery over C in which he receives $a^i \in A$ with probability p_i $(i=1,2,\dots,m)$ and, independently, $b^j \in B$ with probability p_j $(j=1,2,\dots,m)$; and the simultaneous distribution in which he receives (a^i,b^i) with probability p_i : i.e., $[p_ip_j;(a^i,b^j)]_{i,j=1}^{m} \approx_C [p_i;(a^i,b^i)]_{i=1}^{m}$.

(6.3)
$$\begin{aligned}
&[p_i;(a^i,b^i)]_{i=1}^{m}\\
&= [p_1^2\ ;(a^1,b^1),\dots,p_m^2\ ;(a^m,b^m),p_1\,\Sigma_{j\neq 1}\,p_j;(a^1,b^1),\dots,p_m\,\Sigma_{j\neq m}\,p_j;(a^m,b^m)]\\
&= [p_1^2\ ;(a^1,b^1),\dots,p_m^2\ ;(a^m,b^m),2p_1p_2;[\tfrac{1}{2};(a^1,b^1),\tfrac{1}{2};(a^2,b^2)],\\
&\quad 2p_1p_3;[\tfrac{1}{2};(a^1,b^1),\tfrac{1}{2};(a^3,b^3)],\dots,2p_1p_m;[\tfrac{1}{2};(a^1,b^1),\tfrac{1}{2};(a^m,b^m)],\\
&\quad 2p_2p_3;[\tfrac{1}{2};(a^2,b^2),\tfrac{1}{2};(a^3,b^3)],\dots,2p_{m-1}p_m;[\tfrac{1}{2};(a^{m-1},b^{m-1}),\tfrac{1}{2};(a^m,b^m)]]\\
&\approx_C [p_1^2\ ;(a^1,b^1),\dots,p_m^2\ ;(a^m,b^m),2p_1p_2;[\tfrac{1}{2};(a^1,b^2),\tfrac{1}{2};(a^2,b^1)],\dots,\\
&\quad 2p_{m-1}p_m;[\tfrac{1}{2};(a^{m-1},b^m),\tfrac{1}{2};(a^m,b^{m-1})]]\\
&= [p_ip_j\ ;\ (a^i,b^j)]_{i,j=1}^{m}.
\end{aligned}$$

In (6.3), the "$\approx_C$"-step follows from additive independence of $\succ_C$ and (6.1), and the other steps follow from elementary properties of lotteries.

The second condition we need, relates $\succ_C$ to $\succ_A$ and $\succ_B$.

(6.4) There exists a $(a°,b°) \in C$ such that for all $[p_i;a^i]_{i=1}^m$ and $[q_i;\tilde{a}^i]_{i=1}^n$ in $L(A)$, and all $[p_i;b^i]_{i=1}^m$ and $[q_i;\tilde{b}^i]_{i=1}^n$ in $L(B)$, we have :

$$[p_i;a^i]_{i=1}^m >_A [q_i;\tilde{a}^i]_{i=1}^n \Rightarrow [p_i;(a^i,b^\circ)]_{i=1}^m >_C [q_i;(\tilde{a}^i,b^\circ)]_{i=1}^n$$

and

$$[p_i;b^i]_{i=1}^m >_B [q_i;\tilde{b}^i]_{i=1}^n \Rightarrow [p_i;(a^\circ,b^i)]_{i=1}^m >_C [q_i;(a^\circ,\tilde{b}^i)]_{i=1}^n.$$

(*Weak monotonicity*)

The main result of this section is the following variation on Theorem 5.1. in Keeney and Raiffa (1976).

Theorem 6.1. Let A,B, and C=AxB, be sets of alternatives and $\succ_A$, $\succ_B$, and $\succ_C$, preference relations on the corresponding lottery sets, representable by vNM utility functions. Let A and B each contain at least two non-equivalent elements. Then the following two statements are equivalent.

(i) $\succ_C$ satisfies additive independence, and $\succ_A$, $\succ_B$, and $\succ_C$, satisfy weak monotonicity.

(ii) There are vNM utility functions u,v, and w, for $\succ_A$, $\succ_B$, and $\succ_C$, respectively, and positive constants k_u and k_v, such that $w(a,b) = k_u u(a) + k_v v(b)$ for all $(a,b) \in C$.

Proof. The implication (ii) $\Rightarrow$ (i) is straightforward. For (i) $\Rightarrow$ (ii), let $a°$ and $b°$ be as in (6.4), and take $\hat{a} \in A$ and $\hat{b} \in B$ such that $\hat{a} \not\approx_A a°$ and $\hat{b} \not\approx_B b°$. Choose vNM utility functions u,v, and w, for $\succ_A$, $\succ_B$, and $\succ_C$, respectively, such that $u(a°) = v(b°) = w(a°,b°) = 0$ (cf. (4.2)). By weak monotonicity, $w(\hat{a},b°)$ and $u(\hat{a})$ must have the same sign, so $k_u := w(\hat{a},b°)u(\hat{a})^{-1} > 0$. Also, $k_v := w(a°,\hat{b})v(\hat{b})^{-1} > 0$. By additive independence of $\succ_C$ we have for every $(a,b) \in C$

$\frac{1}{2}w(a,b) + \frac{1}{2}w(a°,b°) = \frac{1}{2}w(a,b°) + \frac{1}{2}w(a°,b)$, hence $w(a,b) = w(a,b°) + w(a°,b)$.

The proof is complete if we show : $w(a,b°) = k_u u(a)$ and $w(a°,b) = k_v v(b)$ for every $a \in A$, $b \in B$. We only prove the first equality, we let $a \in A$, and distinguish three cases : $(a°,b°) \succ_C (a,b°)$ and $(\hat{a},b°) \succ_C (a,b°)$; $(a,b°) \succ_C (\hat{a},b°)$ and $(a,b°) \succ_C (a°,b°)$; $(a,b°) \approx_C [\mu;(\hat{a},b°),(1-\mu);(a°,b°)]$ for a (unique) $0 < \mu < 1$ (where such a μ exists since $\succ_C$ is representable by w).

We only consider the third case, the other ones are similar. In that third case, $(a,b°) \approx_C [\mu;(\hat{a},b°),(1-\mu);(a°,b°)]$ implies $a \approx_A [\mu;\hat{a},(1-\mu);a°]$ by weak monotonicity, hence $u(a) = \mu u(\hat{a})$.

So $w(a,b^\circ) = \mu w(\hat{a},b^\circ) = u(a)u(\hat{a})^{-1}w(\hat{a},b^\circ) = k_u u(a)$, which we set out to prove.■

Remark 6.2. Of course, in view of (4.2), we can always rescale u and v in Theorem 6.1 such that $k_u - k_v = 1$. If, in particular, we would have $u(\hat{a}) = v(\hat{b})$ and $(\hat{a},b^\circ) \approx_C (a^\circ,\hat{b})$ in the proof of Theorem 6.1, then $k_u = k_v$, and we may set $k_u = k_v = 1$.

Theorem 6.1 and Remark 6.2 will be used in chapter 6, section 17.

7. MULTIPLICATIVE UTILITY

As in section 6, let A,B, and C := A x B, be sets of alternatives, with corresponding lottery sets L(A), L(B) and L(C), respectively, and preference relations $\succ_A$, $\succ_B$, and $\succ_C$, on these lottery sets. In this section, we shall give necessary and sufficient conditions under which a vNM utility function for $\succ_C$ can be written as a (multiplicative) product of vNM utility functions for $\succ_A$ and $\succ_B$. Again, our result is a variation on a result obtained by Keeney and Raiffa (1976), namely their Theorem 5.2 (see also their section 5.4.3); it was also presented, in a slightly different form, by Binmore (1982), and it will be used in section 12 where we give an extension of an axiomatic bargaining model presented in the last mentioned paper.

Let A,B,C,... be as in the first paragraph. We again assume that $\succ_A$, $\succ_B$, and $\succ_C$, are representable by vNM utility functions. We formulate two conditions for $\succ_A$,$\succ_B$, and $\succ_C$.

(7.1) If $[p_i;a^i]_{i=1}^{n} \approx_A [q_i;\tilde{a}^i]_{i=1}^{m}$ in L(A), and
$[r_i;b^i]_{i=1}^{k} \approx_B [s_i;\tilde{b}^i]_{i=1}^{\ell}$ in L(B), then
$[p_ir_j;(a^i,b^j)]_{i=1,j=1}^{n\ \ k} \approx_C [q_is_j;(\tilde{a}^i,\tilde{b}^i)]_{i=1,j=1}^{m\ \ \ell}$ in L(C).
(*Weak utility independence*)

Condition (7.1) is a weaker version of the "Utility Independence" property in Keeney and Raiffa (1976, p. 224).

(7.2) There exist $\bar{a} \in A$ and $\bar{b} \in B$ such that, for all a,a' ∈ A and b,b' ∈ B, we have : $a \succcurlyeq_A \bar{a}$, $b \succcurlyeq_B \bar{b}$, $(\bar{a},b) \approx_C (\bar{a},b')$, and $(a,\bar{b}) \approx_C (a',\bar{b})$. Furthermore, there exist $a^* \in A$ and $b^* \in B$ such that $a^* \succ_A \bar{a}$, $b^* \succ_B \bar{b}$, and $(a^*,b^*) \succ_C (\bar{a},\bar{b})$.

The first part of (7.2) requires that A and B both have "worst" alternatives, which make all combinations in C in which they occur, equivalent. The "furthermore"-part of (7.2) serves to avoid trivality.

With these conditions, we can state and prove the following theorem.

Theorem 7.1. With notations as in the first paragraph of this section, and under the assumption of the second paragraph, the following two statements are equivalent.

(i) (7.1) and (7.2) hold.

(ii) There exist vNM utility functions w,u, and v, for $\succcurlyeq_C$, $\succcurlyeq_A$, and $\succcurlyeq_B$, respectively, and $\bar{a},a^* \in A$, $\bar{b},b^* \in B$, such that for all $a \in A$ and $b \in B$ we have : $w(a,b) = u(a)v(b) \geq 0$, $u(\bar{a}) = v(\bar{b}) = 0$, $u(a^*) > 0$, $v(b^*) > 0$.

Proof. The implication (ii) $\Rightarrow$ (i) is straightforward. For (i) $\Rightarrow$ (ii), let $\bar{a},\bar{b},a^*,b^*$ be as in (7.2). Let w,u,v be vNM utility functions for $\succcurlyeq_C$, $\succcurlyeq_A$, $\succcurlyeq_B$, such that : $u(\bar{a}) = v(\bar{b}) = w(\bar{a},\bar{b}) = 0$, and $w(a^*,b^*) = u(a^*)v(b^*) > 0$. Let $a \in A$, $b \in B$. The proof is finished if we show : $w(a,b) = u(a)v(b)$.

We first calculate $w(a,b^*)$. There are two cases : $a \approx_A [\mu;\bar{a},(1-\mu);a^*]$ for some $0 \leq \mu \leq 1$; and $a^* \approx_A [\mu;\bar{a},(1-\mu);a]$ for some $0 < \mu < 1$. We only consider the first case, the second one is similar. In that case, by (7.1), we have $w(a,b^*) = Ew([\mu;(\bar{a},b^*),(1-\mu);(a^*,b^*)])$, so in view of (7.2) :

$w(a,b^*) = (1-\mu)w(a^*,b^*) = (1-\mu)u(a^*)v(b^*) = u(a)v(b^*)$.

Finally, we calculate $w(a,b)$. Again : either $b \approx_B [\mu;\bar{b},(1-\mu);b^*]$ for some $0 \leq \mu \leq 1$; or $b^* \approx_B [\mu;\bar{b},(1-\mu);b]$ for some $0 < \mu < 1$. And again, we only consider the first case. Then, by (7.1), we have

$w(a,b) = Ew([\mu;(a,\bar{b}),(1-\mu);(a,b^*)])$, so

$w(a,b) = (1-\mu)w(a,b^*) = (1-\mu)u(a)v(b^*) = u(a)v(b)$.■

BARGAINING SITUATIONS, GAMES, SOLUTIONS

This chapter introduces three basic concepts : bargaining situations (in section 8), bargaining games (in section 9), and bargaining solutions (in section 10). These concepts will be defined for the general n-person case.

8. BARGAINING SITUATIONS

Bargaining situations may arise in case of a conflict between several parties, such that unanimity is required in order to solve the conflict. In what follows, $n \in \mathbb{N}$ with $n \geq 2$, always denotes the number of such parties (players).

Definition 8.1. An *n-person bargaining situation* is an (n+2)-tuple

$$\Gamma := \langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle$$

where

(8.1) A is a compact subset of $\mathbb{R}^m$, for some $m \in \mathbb{N}$,

(8.2) for every $i = 1,2,\ldots,n$, u^i is a continuous element of $U(A)$,

(8.3) $\bar{a} \in A$ such that $u^i(\bar{a}) = 0$, and there exists an $\ell \in L(A)$ with $Eu^i(\ell) > 0$, for all $i = 1,2,\ldots,n$.

As before, A is called the *set of riskless alternatives*. Compactness of A and continuity of the u^i are required for the convenience of $u(A)$ being a compact subset of $\mathbb{R}^n$: cf. the next section. Here, we denote by u the n-tuple $(u^1, u^2, \ldots, u^n)$, so $u(A) = \{(u^1(a), u^2(a), \ldots, u^n(a)) : a \in A\}$. The element $\bar{a}$ is called the *disagreement alternative*.

The game-theoretic interpretation of an n-person bargaining situation $\Gamma = \langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle$ is as follows. There are n *players* or *bargainers*, called $1,2,\ldots,n$, bargaining over the lottery set $L(A)$. They may either reach an agreement $\ell \in L(A)$, giving utility $Eu^i(\ell)$ to player i, or fail to do so, which is the case if no unanimous decision on some lottery can be attained : in other words, each player has a veto right. In the latter case, the bargaining situation ends in the disagreement (or status quo) alternative $\bar{a}$, giving utility $0 = u^i(\bar{a})$ to player i. Every u^i is considered to be a von Neumann - Morgenstern utility function (for some preference relation on $L(A)$) normalized such that $u^i(\bar{a}) = 0$ (cf. section 4, esp. (4.2)). The second part of condition

(8.3) is included to give each player an incentive to cooperate.

We collect a few useful notations in :

Definition 8.2. By BS^n, we denote the family of all n-person bargaining situations. We shall often use BS instead of BS^2. Finally, we denote by N the set of players $\{1,2,\ldots,n\}$.

We conclude this section with a few examples.

Example 8.3. Two players are bargaining over a bottle of wine which must remain closed, that is, divisions of the content of the bottle are not allowed. To each player, [not] obtaining the bottle represents a utility of 1[0]. Disagreement means, that no player gets the bottle. This situation can be modelled as a 2-person bargaining situation $\Gamma = \langle A,\bar{a},u^1,u^2 \rangle$ where : $A = \{(1,0),(0,1),(0,0)\}$ with $(1,0)$ $[(0,1)]$ the alternative that player 1[2] gets the bottle; $\bar{a} = (0,0)$ the disagreement alternative; $u^1(\bar{a}) = u^2(\bar{a}) = u^1((0,1)) = u^2((1,0)) = 0$ and $u^1((1,0)) = u^2((0,1)) = 1$.

Example 8.4. We consider the same situation as in Example 8.3, but now with any division of the content of the bottle of wine allowed. Each player's utility is assumed to be proportional to the amount of wine that player obtains. So this situation can be described by $\Gamma = \langle A,\bar{a},u^1,u^2 \rangle \in BS$ where : $A = \{\bar{a}\} \cup \{(x,1-x) : x \in [0,1]\}$ with $(x,1-x)$ meaning that player 1 (respectively 2) gets 100x(respectively 100-100x) percent of the wine; $u^1(\bar{a}) = u^2(\bar{a}) = 0$ and $u^1((x,1-x)) = u^2((1-x,x)) = x$ for every $x \in [0,1]$.

Example 8.5. We consider the same situation as in Example 8.4, but with the players' utility functions now satisfying : $u^1(\bar{a}) = u^2(\bar{a}) = 0$, $u^1((x,1-x)) = x$ and $u^2((1-x,x)) = \sqrt{x}$ for every $x \in [0,1]$.

Examples 8.3 - 8.5 will return in the following section.

9. BARGAINING GAMES

For a non-empty set $S \subset \mathbb{R}^n$, we call

$$\mathrm{com}(S) := \{x \in \mathbb{R}^n : x \leq s \text{ for some } s \in S\}$$

the *comprehensive hull of* S. We call S *comprehensive* if $S = \text{com}(S)$. Further, $\text{comv}(S) := \text{com}(\text{conv}(S))$ is called the *comprehensive convex hull of* S. (It is not hard to prove that $\text{com}(\text{conv}(S)) = \text{conv}(\text{com}(S))$.)

A "bargaining game" may arise as the comprehensive hull of the image in expected utility space of the set of lotteries in a bargaining situation, or, equivalently, as the comprehensive convex hull of the image in utility space of the set of riskless alternatives in a bargaining situation. First, however, we give the general definition. As before, $n \in \mathbb{N}$, $n \geq 2$.

Definition 9.1. An *n-person bargaining game* S is a convex closed comprehensive subset of $\mathbb{R}^n$ which has a strictly positive element and for which $\max\{s_i : s \in S\}$ exists for every $i \in \mathbb{N}$.

Elements of S are called *outcomes*, and 0 is called the *disagreement outcome* or *disagreement point*. The game-theoretic interpretation of a bargaining game S is similar to the interpretation of a bargaining situation in the previous section. Closedness and comprehensiveness of S are required mainly for mathematical convenience; comprehensiveness, moreover, can be interpreted as the possibility of free disposal of utility. An explanation of the convexity of S is given by the possibility of an underlying bargaining situation, as will become clear in what follows.

As in Nash (1950), where bargaining games were introduced, it is often assumed in the bargaining literature, that S is compact. For most of the results in this monograph, it would not make any essential difference if compactness of a bargaining game were required instead of comprehensiveness; if, nevertheless, that does make an important difference, then we will explicitly mention it.

Another deviation from the standard definition of a bargaining game is, that we always take the origin as disagreement outcome; the reason for this will become clear in the next section, where we discuss bargaining solutions.

Definition 9.2. By B^n we denote the family of all n-person bargaining games. We will often write B instead of B^2.

The following definition assigns a bargaining game to each bargaining situation.

Definition 9.3. Let $\Gamma = \langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle \in BS^n$. We call $S_\Gamma := \text{com}(Eu(L(A))) = \text{comv}(u(A))$ the *bargaining game corresponding to* Γ. (Recall that $u = (u^1, u^2, \ldots, u^n)$.)

Note that, indeed, $S_\Gamma \in B^n$ (since, in particular, $u(A) \subset \mathbb{R}^n$ is compact). Conversely, for each bargaining game there is a bargaining situation to which it correponds, as the next example shows. First, we give a few definitions.

Definition 9.4. For every $i \in N$, the function $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is defined by $\pi^i(x) := x_i$ for each $x \in \mathbb{R}^n$. So π^i is the projection on the i-th coordinate.

Definition 9.5. For every non-empty set $S \subset \mathbb{R}^n$, we denote by

$$P(S) := \{s \in S : \text{if } x \in S \text{ and } x \geq s, \text{ then } x = s\}$$

the *Pareto optimal subset of* S.
(Note that, if $S \in B^n$, then $P(S)$ is bounded because $\max\{s_i : s \in S\}$ exists for every $i \in N$.)

Example 9.6. Let $S \in B^n$. The *trivial bargaining situation corresponding to* S is the bargaining situation $\Gamma = \langle A, \bar{a}, u^1, u^2, \ldots, u^n \rangle$ where $A := \{0\} \cup \text{cl}(P(S))$, $\bar{a} = 0 \in \mathbb{R}^n$, $u^i(a) = \pi^i(a) = a_i$ for every $i \in N$ and $a \in A$. (By "cl" we denote (topological) closure.)

The following three examples correspond to Examples 8.3 - 8.5.

Example 9.7. Consider the 2-person bargaining situation Γ as defined in Example 8.3. Then $S_\Gamma = \text{comv}(\{(0,0),(1,0),(0,1)\})$.

Example 9.8. Consider $\Gamma \in BS$ as in Example 8.4. Then $S_\Gamma = \text{comv}(\{(0,0),(1,0),(0,1)\})$.

Example 9.9. Consider $\Gamma \in BS$ as in Example 8.5. Then $S_\Gamma = \text{com}(\{(x, \sqrt{1-x}) : 0 \leq x \leq 1\})$.

Definition 9.10. Let $S \in B^n$. We call

$$\partial S = W(S) := \{s \in S : \text{there is no } x \in S \text{ with } x > s\}$$

the *boundary* or *weakly Pareto optimal subset of* S. We call $g(S) \in \mathbb{R}^n$ with

$g_i(S) := \max\{s_i : s \in S\}$ for every $i \in N$, the *global utopia point of* S. We denote $S_+ := \mathrm{com}(S \cap \mathbb{R}^n_+)$, and call $h(S) := g(S_+)$ the *utopia point of* S. By B^n_+, we denote the family of n-person bargaining games S with $S = S_+$. We write also B_+ instead of B^2_+.

Definition 9.11. For every $i \in N$, the function $\pi_{-i} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is defined by $x \mapsto (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Definition 9.12. Let $S \in B^n$. For every $i \in N$, the *Pareto function* $f^i_S : \pi_{-i}(S) = \{\pi_{-i}(s) : s \in S\} \to \mathbb{R}$ is defined by

$$\pi_{-i}(s) \mapsto \max\{x_i : (s_1, s_2, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n) \in S\}.$$

Some properties of Pareto functions are collected in the following lemma, the proof of which is elementary and left to the reader.

Lemma 9.13. Let $S \in B^n$. Then :

(i) For every $i \in N$, f^i_S is a concave function.

(ii) The union of the graphs of the functions f^i_S is equal to $\partial S = W(S)$.

(iii) The intersection of the graphs of the functions f^i_S is equal to $P(S)$.

(iv) If n=2, then f^1_S and f^2_S are nonincreasing functions.

10. BARGAINING SOLUTIONS

In the bargaining literature, an n-person bargaining game $S \in B^n$ is sometimes called a "bargaining problem", see for instance Nash (1950) (for n=2). Let us use the expression "bargaining problem" for the following problem : which agreement (outcome) will or should the players in a bargaining game S reach, if any ? Nash (1950) proposed to consider not each bargaining game separately, but to look at the family of all bargaining games (2-person, in his case). Formally :

Definition 10.1. Let $C^n \subset B^n$, $n \geq 2$. An (n-*person*) *bargaining solution for* (or : *on*) C^n is a map $\phi : C^n \to \mathbb{R}^n$ with $\phi(S) \in S$ for every $S \in C^n$.

As far as the context allows and confusion is unlikely we will use the expressions : n-person bargaining solution, bargaining solution, (n-person) solution, and omit "for C^n" whenever it is clear which class is meant.

We always - by definition - take the origin as disagreement outcome of a bargaining game. Alternatively, we could have considered only "translation invariant" bargaining solutions in this monograph. This "translation invariance" property is part of a property often called "independence of equivalent utility transformations (IEUT)" (see, e.g. Roth (1979)); the latter property reflects (4.2) in case the use of vNM utility functions is assumed. The other part of "IEUT" is the so-called "scale transformation invariance" property (see Def. 10.5 below), which will frequently be used in this monograph.

<u>Definition 10.2.</u> Let $C^n \subset B^n$. We denote by $CS^n \subset BS^n$ the family of bargaining situations Γ in BS^n such that $S_\Gamma \in C^n$. For instance, if $C^n = B^n_+$, then $B_+S^n = \{\Gamma \in BS^n : S_\Gamma \in B^n_+\}$.

<u>Definition 10.3.</u> Let ϕ be a solution for $C^n \subset B^n$. We call the map $\tilde{\phi} : CS^n \to \mathbb{R}^n$ defined by $\tilde{\phi}(\Gamma) := \phi(S_\Gamma)$, the *bargaining situation solution corresponding to* ϕ. Mostly, we write ϕ instead of $\tilde{\phi}$. We further denote, for $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n \rangle \in CS^n$:

(10.1) $\mathrm{alt}(\phi,\Gamma) := \{\ell \in L(A) : Eu(\ell) = \phi(\Gamma)\}$

and we call this set the *set of ϕ-alternatives of* Γ.

Thus, we let a bargaining situation solution depend exclusively on the bargaining game corresponding to a bargaining situation. In the larger part of this monograph, we only consider bargaining games. Bargaining situations become important especially when we study risk properties of solutions, in chapter 8 and in section 30. Abstracting from underlying bargaining situations as is done in the definition of $\tilde{\phi}$ above, may lead to "paradoxes" : we refer, in particular, to Shapley (1969). As a final remark concerning Def. 10.3, we note that the set of ϕ-alternatives of Γ may be empty, which is due to the fact that S_Γ is the comprehensive hull of (and hence, larger than) $Eu(L(A))$.

In this monograph, we will be mainly concerned with the study of relations between properties of bargaining solutions. Some of these properties are fairly standard and frequently used. Therefore we introduce them already here.

For $x,y \in \mathbb{R}^n$ and $T \subset \mathbb{R}^n$, we denote $xy := (x_1y_1,x_2y_2,\ldots,x_ny_n) \in \mathbb{R}^n$ and $xT := \{xy : y \in T\}$.

Definition 10.4. We call a bargaining solution $\phi : B^n \to \mathbb{R}^n$ *homogeneous* if $\phi(\lambda S) = \lambda\phi(S)$ for every $\lambda \in \mathbb{R}$ with $\lambda > 0$ and every $S \in B^n$.
(Homogeneity : HOM)

Definition 10.5. We call a bargaining solution $\phi : B^n \to \mathbb{R}^n$ *scale transformation invariant* if $\phi(aS) = a\phi(S)$ for every $S \in B^n$ and every $a \in \mathbb{R}^n_{++}$. (In this context, we call such an $a \in \mathbb{R}^n_{++}$ a *scale transformation*.)
(Scale transformation invariance : STI)

The STI-property has been mentioned before (see after Def. 10.1) and is a natural consequence of the use of vNM utility functions in case underlying bargaining situations are assumed. Homogeneity is implied by scale transformation invariance.

Definition 10.6. We call a bargaining solution $\phi : B^n \to \mathbb{R}^n$ *weakly Pareto optimal* if $\phi(S) \in W(S)$, *Pareto optimal* if $\phi(S) \in P(S)$, *individually rational* if $\phi(S) \geqq 0$, and *strongly individually rational* if $\phi(S) > 0$, for every $S \in B^n$.
(Weak Pareto optimality : WPO; Pareto optimality : PO; individual rationality : IR; strong individual rationality : SIR).

The properties in Def. 10.6 need no further explanation. The final two properties we introduce, require a bargaining solution to be independent of the identities of the players, in two senses to be specified. Some notations : for $x \in \mathbb{R}^n$, $T \subset \mathbb{R}^n$, and a permutation $\pi : N \to N$, we denote $\pi x := (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ and $\pi T := \{\pi x : x \in T\}$.

Definition 10.7. We call a bargaining solution $\phi : B^n \to \mathbb{R}^n$ *anonymous* if $\phi(\pi(S)) = \pi\phi(S)$ for every $S \in B^n$ and every permutation π of N. We call ϕ *symmetric* if $\phi_1(S) = \phi_2(S) = \ldots = \phi_n(S)$ for every *symmetric* S in B^n, i.e. every S with $S = \pi(S)$ for every permutation π of N.
(Anonymity : AN; symmetry : SYM)

SYM is weaker than AN :

Example 10.8. Let the 2-person solution $\phi : B \to \mathbb{R}^2$ be defined by :

$$\phi(S) := \begin{cases} \text{the point in } P(S) \cap \mathbb{R}^2_+ \text{ where the product } x_1x_2 \text{ is maximized, if} \\ S \in B \text{ is such that } aS \text{ is symmetric for some } a \in \mathbb{R}^2_{++} \\ \text{the point in } P(S) \cap \mathbb{R}^2_+ \text{ with maximal first coordinate, otherwise.} \end{cases}$$

This ϕ satisfies IR, PO, STI, SYM, but not AN.

Nonsymmetric solutions play an important role in this monograph. An advantage of the consideration of nonsymmetric solutions is, that, whenever external factors exist which enable us to distinguish between the identities of the players, we have a theory for such cases at hand. An example of this is the market model in section 13.

All properties in this section are stated for solutions on B^n. Of course, the same properties can, in general, be defined for the restrictions of such solutions to subclasses of B^n, or for solutions defined only for such subclasses. If no ambiguities are likely to arise, we will omit such definitions.

INDEPENDENCE OF IRRELEVANT ALTERNATIVES AND RELATED PROPERTIES

Axiomatic bargaining theory started with the paper of Nash (1950). His (2-person) solution concept was later (Harsanyi and Selten (1972)) extended to a family of nonsymmetric solutions. In the first section of this chapter (section 11) we shall give a further extension of these results. Section 12 provides two other characterizations of the same family of solutions. In section 13, a market model is proposed, which, in particular, accounts for the nonsymmetry of solutions in this family. The remaining section 14 in this chapter gives another model which gives rise to Nash solutions, and briefly mentions some literature.

11. INDEPENDENCE OF IRRELEVANT ALTERNATIVES

Nash (1950) has proposed the following property for 2-person bargaining solutions.

Definition 11.1. We call a 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ *independent of irrelevant alternatives* if for all S and T in B with $S \subset T$ and $\phi(T) \in S$, we have $\phi(S) = \phi(T)$.
(Independence of irrelevant alternives : IIA)

Another way to formulate the IIA-property is : if a bargaining game S is enlarged to T, then either the solution outcome remains the same, or it becomes one of the newly added outcomes. There has been much discussion on this property in the literature, some of which has led to the proposal of a new property : see section 15.

Definition 11.2. For $S \in B$, we denote by $\underline{p}(S)$ the point in $P(S_+)$ with maximal first coordinate, and by $\bar{p}(S)$ the point in $P(S_+)$ with maximal second coordinate. (Note that $P(S_+) = P(S) \cap \mathbb{R}^2_+$.)

Definition 11.3. For every $0 \leq t \leq 1$, we define the solution $N^t : B \to \mathbb{R}^2$ as follows. If $0 < t < 1$, then, for every $S \in B$, $N^t(S)$ is the unique point of $P(S_+)$ where the product $x_1^t x_2^{1-t}$ is maximized. $N^0(S)$ and $N^1(S)$ are the points in $P(S_+)$ where x_2 and x_1, respectively, are maximized. So $N^0(S) = \bar{p}(S)$ and $N^1(S) = \underline{p}(S)$ for every $S \in B$. We also use the notations D^1 and D^2 for N^1 and N^0, respectively, and call these solutions the *player* 1 and *player* 2 *dictator solutions*, respectively. Every solution N^t is called *a* (*nonsymmetric*) *Nash solution*, and the solution $N^{\frac{1}{2}}$, also denoted N, is called *the* (*symmetric*) *Nash solution*.

The Nash solution was introduced in Nash (1950). Nash proved the following theorem.

Theorem 11.4. The solution $N : B \to \mathbb{R}^2$ is the unique 2-person bargaining solution with the properties WPO, STI, SYM, and IIA.

Of course, in this theorem, the WPO property may be replaced by the stronger PO property. The solutions N^t ($t \in (0,1)$) were introduced in Harsanyi and Selten (1972).

A proof of the following theorem can be found, e.g., in Roth (1979).

Theorem 11.5. A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ has the properties SIR,STI, and IIA, if and only if $\phi = N^t$ for some $0 < t < 1$.

An immediate consequence of Theorem 11.5 is, that SIR, STI, and IIA, together imply PO. The following lemma gives a direct proof of this fact.

Lemma 11.6. Let the solution $\phi : B \to \mathbb{R}^2$ have the properties SIR, STI, and IIA. Then ϕ is Pareto optimal.
Proof. Let $S \in B, x \in S, x \geqq \phi(S)$. Let $a \in \mathbb{R}^2_{++}$ be such that $ax = \phi(S)$. Such an a exists by SIR. Then $a \leqq (1,1)$, so $aS \subset S$; since $\phi(S) \in aS$, we obtain by IIA : $\phi(aS) = \phi(S)$, so by STI : $a = (1,1)$. Hence $x = \phi(S)$, from which we conclude that ϕ is Pareto optimal. ∎

A consequence of Lemma 11.6 is, that in Theorem 11.4, the WPO property may be replaced by SIR. Roth (1977) shows, that if we replace, in Theorem 11.4, WPO by IR, than there are exactly two solutions satisfying the four properties, namely N, and the solution given by the following definition.

Definition 11.7. The *disagreement solution* D assigns to every $S \in B$ the disagreement outcome 0.

More variations on this theme are possible. In chapter 7, we will study weakly Pareto optimal (multi)solutions , and also "probabilistic solutions", which satisfy some version of the IIA property. In this chapter, we confine further attention to the family of solutions characterized in the following theorem (cf. de Koster et al. (1983)).

Theorem 11.8. $\{N^t : 0 \leq t \leq 1\}$ is the family of all 2-person bargaining solutions for B with the properties IR, PO, STI, and IIA.

Proof. (See Fig. 11.1.) Let $\phi : B \to \mathbb{R}^2$ be a 2-person solution. If $\phi = N^t$ for some $0 \leq t \leq 1$, then it is straightforward to verify that ϕ has the four properties in the theorem.

Suppose, now, that ϕ has these four properties. If $\phi = N^t$ for some $0 < t < 1$, then the proof is finished. So suppose this is not the case, then it follows from Theorem 11.5 that ϕ cannot satisfy SIR. Hence, there must be a game $S \in B$ such that $\phi(S) = \bar{p}(S)$ with $\bar{p}_1(S) = 0$ or $\phi(S) = \underline{p}(S)$ with $\underline{p}_2(S) = 0$.

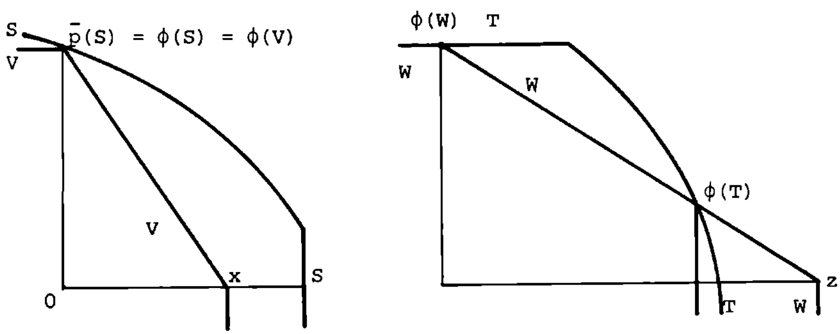


Figure 11.1.

W.l.o.g. suppose the former is the case (the other case is similar, leading to $\phi = N^1$). Take $x \in S$ with $x_2 = 0$ and $x_1 > 0$, and let $V := \mathrm{comv}(\{\bar{p}(S), x\})$. By IIA, $\phi(V) = \bar{p}(S)$. By STI then, $\phi(\Delta) = (0,1)$ where $\Delta := \mathrm{comv}(\{(1,0),(0,1)\}) \in B$. Suppose there exists T with $\phi(T) \neq \bar{p}(T) = N^0(T)$. We derive a contradiction and may conclude then that $\phi = N^0$. Since $\phi(T) \neq \bar{p}(T)$, we have that $\phi_2(T) < \bar{p}_2(T)$. Let z be the point of intersection with the utility axis for player 1 of the straight line through $\phi(T)$ and $(0,\bar{p}_2(T))$. Let $W := \mathrm{comv}(\{(0,\bar{p}_2(T)), z\})$. Then, by STI and $\phi(\Delta) = (0,1)$, we obtain $\phi(W) = (0,\bar{p}_2(T))$, and hence, by IIA, $\phi(\mathrm{comv}(\{(0,\bar{p}_2(T)),\phi(T)\})) = (0,\bar{p}_2(T))$. On the other hand, also by IIA, $\phi(\mathrm{comv}(\{(0,\bar{p}_2(T)),\phi(T)\})) = \phi(T)$, a contradiction since $\phi_2(T) < \bar{p}_2(T)$. ■

The next section provides two other characterizations of the family of nonsymmetric Nash solutions; in each one of them, some other property replaces IIA.

12. INDEPENDENCE OF IRRELEVANT EXPANSIONS AND CONVENTION CONSISTENCY

This section consists of two parts. In the first part, part A, we give a characterization of the family of nonsymmetric Nash solutions $\mathcal{N} := \{N^t : 0 \leq t \leq 1\}$, with the aid of a property called "Independence of irrelevant expansions". Thereby we extend a result of Thomson (1981).

In the second part, part B, we give another characterization of $\mathcal{N}$ with the aid of a property called "Convention consistency" : here, we extend a result of Binmore (1982), and apply Theorem 7.1.

Before starting with part A, we give a lemma which gives a geometric characterization of a solution N^t, for $0 < t < 1$.

<u>Lemma 12.1.</u> Let $S \in B$, $t \in (0,1)$, $z \in P(S)$. Then $N^t(S) = z$ if and only if the straight line with equation $\frac{t}{z_1}x_1 + \frac{1-t}{z_2}x_2 = 1$ is a supporting line of S at z.

<u>Proof.</u> Firstly, let $N^t(S) = z$. Then z is the (unique) point of intersection of S and the set $H := \{x \in \mathbb{R}^2_{++} : x_1^t x_2^{1-t} \geq z_1^t z_2^{1-t}\}$ (uniqueness of z was pre-assumed, but follows of course from the convexity of S and the strict convexity of H). So there exists a straight line through z separating S and H. The equation of this line can be found by implicit differentiation, with the aid of

the equation of ∂H.

Secondly, suppose that at some $z \in P(S)$ we have a supporting line of S with equation as in the lemma. Then, again by differentiating, it follows that this line also supports H at z, which implies that $x_1^t x_2^{1-t}$ is maximal on $S \cap \mathbb{R}^2_+$ in z, and hence $z = N^t(S)$. ■

Shapley (1969) offers an interpretation of the (symmetric) Nash solution $N = N^{\frac{1}{2}}$ in terms of a compromise between "maximization of social welfare" and "sharing of social profit", which is based on the geometric characterization in Lemma 12.1, for $t = \frac{1}{2}$.

Part A : Independence of irrelevant expansions

For $x,y \in \mathbb{R}^n$, we denote by $x.y$ the inner product $\Sigma_{i=1}^n x_i y_i$.

Definition 12.2. (Cf. Fig. 12.1.) We call a solution $\phi : B \to \mathbb{R}^2$ *independent of irrelevant expansions* if for every $S \in B$ there exists a $p \in \mathbb{R}^2_{++}$ with $p_1 + p_2 = 1$ such that :

(i) $p.x = p.\phi(S)$ is the equation of a supporting line of S at $\phi(S)$,

(ii) for all $T \in B$ with $S \subset T$ and $p.x \leq p.\phi(S)$ for all $x \in T$, we have $\phi(T) \geq \phi(S)$.

(Independence of irrelevant expansions : IIE)

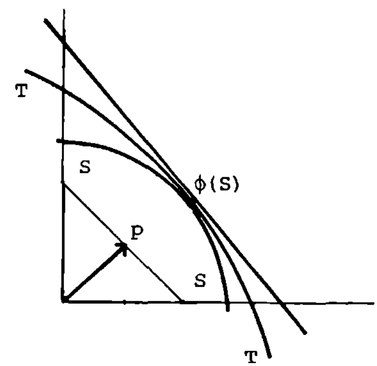


Figure 12.1.

We call an $S \in B$ *smooth at* $x \in \partial S$ if S has a unique supporting line at x.

Lemma 12.3. Let $\phi : B \to \mathbb{R}^2$ satisfy IIE, let S be smooth at $\phi(S)$, and let $T \in B$ with $S \subset T$ and $\phi(S) \in P(T)$. Then $\phi(T) = \phi(S)$.

Proof. Because $\phi(S) \in P(T)$, the proof is complete if we have shown that (ii) of Def. 12.2 holds, with $p.x = p.\phi(S)$, where $p \geq 0$ and $p_1 + p_2 = 1$, being the equation of the unique supporting line of S at $\phi(S)$. Suppose, contrary to what we want to prove, that $p.t > p.\phi(S)$ for some $t \in T$. Since $t \notin S$ and $\phi(S) \in P(T)$, we may suppose w.l.o.g. : $t_1 < \phi_1(S)$, $t_2 > \phi_2(S)$. Let $q \geq 0$, with $q_1 + q_2 = 1$, such that $q.t = q.\phi(S)$. Then $q_1 > p_1$ and $q_2 < p_2$, because $q_2(t_2 - \phi_2(S)) = q_1(\phi_1(S) - t_1)$ whereas $p_2(t_2 - \phi(S)) > p_1(\phi_1(S) - t_1)$. A similar argument gives : if $x \in \mathbb{R}^2$ with $x_1 < \phi_1(S)$, $x_2 > \phi_2(S)$, and $q.x > q.\phi(S)$, then $p.x > p.\phi(S)$, so $x \notin S$. Hence, if $s \in S$ with $q.s > q.\phi(S)$, then $s_1 > \phi_1(S)$ and $s_2 < \phi_2(S)$. Take such an s, say $\bar{s}$: this $\bar{s}$ exists since $q.x = q.\phi(S)$ cannot be the equation of a supporting line of S. Then there is an $\alpha > 0$ such that $\alpha\bar{s} + (1-\alpha)t > \phi(S)$, a contradiction since $\phi(S) \in P(T)$ and $\alpha\bar{s} + (1-\alpha)t \in T$. ∎

Lemma 12.3 shows that, under the additional smoothness assumption, there is a close relationship between the formulations of IIE and IIA. The main result of the present part A of this section will be that in Theorem 11.8, IIE can replace IIA. (Thomson (1981) shows the same for Theorem 11.4.) The remainder of part A is devoted to the proof of this assertion. We will use the notation $p(\phi,S)$ for p in Def. 12.2.

Lemma 12.4. Let $v,w \in \mathbb{R}^2$ with $v_1,w_2 > 0$ and $v_2,w_1 < 0$ and such that $0 \in \text{int}(V)$ where $V := \text{comv}(\{v,w\})$. Let $\phi : B \to \mathbb{R}^2$ be a solution satisfying IR, PO, STI, and IIE, and such that $\phi_1(V) > 0$. Let $z \in P(V)$, $z \geq 0$, $z_1 < \phi_1(V)$, and $y \in \text{int}(V)$ with $w_1 < y_1 < 0$ and $z_2 < y_2$. Finally, let $T := \text{comv}(\{v,z,y\})$, and $W := \text{comv}(\{v,u\})$ where $u \in P(V)$ with $u_2 = y_2$. Then $\phi(T) = \phi(V) = \phi(W)$.

Proof. (See Fig. 12.2.) First we show that $\phi(T) = \phi(V)$.
Suppose that $\phi_1(T) \leqq z_1$. If $\phi_1(T) < z_1$, then, necessarily, $p_1(\phi,T) < p_1(\phi,V)$, and if $\phi(T) = z$, then also $p_1(\phi,T) < p_1(\phi,V)$, since otherwise we would contradict IIE applied to $T \subset V$. Now perform the scale transformation $a \in \mathbb{R}^2_{++}$ on V such that $p(\phi,aV) = p(\phi,T)$ and $z \in P(aV)$. It follows that $a_2 < 1$, and $T \subset aV$. Further, $p(\phi,T).x = p(\phi,aV).x \leqq p(\phi,aV).\phi(T)$ for all $x \in aV$. So by IIE,

$\phi(aV) \geqq \phi(T)$, hence by STI, $a\phi(V) \geqq \phi(T)$, which contradicts $a_2 < 1$ and $\phi_2(V) < \phi_2(T)$.

So we must have $\phi_1(T) > z_1$, and hence by Lemma 12.3 applied to $T \subset V$: $\phi(T) = \phi(V)$.

Finally, by Lemma 12.3 applied to $T \subset W$, we obtain $\phi(T) = \phi(W)$. ∎

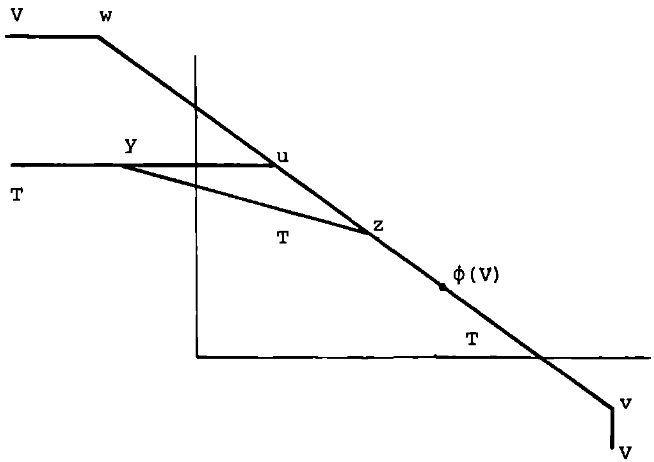


Figure 12.2.

Theorem 12.5. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution. Then ϕ satisfies IR, PO, STI, and IIE, if and only if $\phi \in N$.

Proof. First, suppose $\phi \in N$. By Theorem 11.8, ϕ satisfies IR, PO, and STI. Take $S \in B$. If $\phi = N^1$, then ϕ satisfies IIE with $p(\phi,S) = (1,0)$ if $N_2^1(S) > 0$ and with for $p(\phi,S)$, e.g., the vector with maximal first coordinate among all normal vectors of length 1 of supporting lines of S at $N^1(S)$, otherwise. Analogously for $\phi = N^0$. If $\phi = N^t$ for some $t \in (0,1)$, then, as a consequence of Lemma 12.1, ϕ satisfies IIE with $p(\phi,S)$ being a multiple of the vector $(tN_1^t(S)^{-1},(1-t)N_2^t(S)^{-1})$.

Secondly, suppose ϕ satisfies IR, PO, STI, and IIE. In view of Theorem 11.8, the proof is complete if we show that ϕ satisfies IIA. So let now S and T in B with $S \subset T$ and $\phi(T) \in S$. We have to prove :

(12.1) $\phi(S) = \phi(T)$.

Suppose, to the contrary, that $\phi(S) \neq \phi(T)$. W.l.o.g. assume $\phi_1(S) < \phi_1(T)$. We distinguish two cases : (a) $p(\phi,T) = (1,0)$; (b) $p(\phi,T) > 0$ ($p(\phi,T) = (0,1)$ is excluded by PO).

(a) $p(\phi,T) = (1,0)$. Note that $p(\phi,S) \neq (1,0)$. If $p(\phi,S) = (0,1)$, then take $x,y \in \mathbb{R}^2_{++}$ with $x_2 = \phi_2(S)$ and $\phi_1(S) < x_1 < \phi_1(T)$, $y_1 = \phi_1(T)$ and $\phi_2(T) < y_2 < \phi_2(S)$. (Cf. Fig. 12.3(a).) Then $S \subset \text{comv}(\{x,y\})$. By IIE and PO applied to $S \subset \text{comv}(\{x,y\})$, we obtain $\phi(\text{comv}(\{x,y\})) = x$, hence $\phi_1(\text{comv}(\{x,y\})) < \phi_1(T)$. On the other hand, we can perform a scale transformation $(1,a_2) \in \mathbb{R}^2_{++}$ with $a_2 > 1$ on $\text{comv}(\{x,y\})$ such that $T \subset (1,a_2)\text{comv}(\{x,y\})$ and, then, by IIE, $\phi((1,a_2)\text{comv}(\{x,y\})) \geq \phi(T)$, contradicting, by STI, $\phi_1(\text{comv}\{x,y\}) < \phi_1(T)$.

Next, let $p(\phi,S) > 0$. (Cf. Fig. 12.3(b).) Denote by ℓ the straight line with equation $p(\phi,S).x = p(\phi,S).\phi(S)$, and take points v and w on this line as in Lemma 12.4, and such that $S \subset V := \text{comv}(\{v,w\})$. By IIE, $\phi(V) = \phi(S)$. By Lemma 12.4 (interchanging there the roles of the players), we then obtain $\phi(W) = \phi(V) = \phi(S)$ where $W := \text{comv}(\{w,x\})$ and $x \in \ell$ with $\phi_1(S) < x_1$ and $x_2 > 0$. Apply a scale transformation $(x_1^{-1}\,\phi_1(T),a_2) \in \mathbb{R}^2_{++}$ on W with $a_2 > 1$ such that $T \subset (x_1^{-1}\,\phi_1(T),a_2)W$. By IIE and STI,

$x_1^{-1}\,\phi_1(T)\phi_1(W) \geq \phi_1(T)$, hence $\phi_1(W) \geq x_1$, and so $\phi_1(S) \geq x_1$, a contradiction. So for the case $p(\phi,T) = (1,0)$, we have proved (12.1).

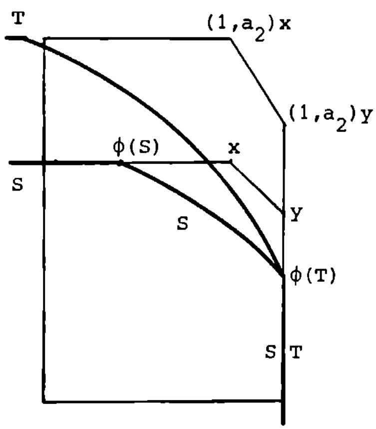


Figure 12.3(a) :
$p(\phi,S) = (0,1)$

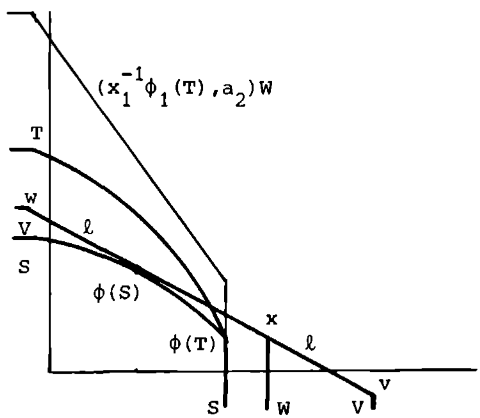


Figure 12.3(b) :
$p(\phi,S) > 0$

(b) $p(\phi,T) > 0$. Let now ℓ be the straight line with equation $p(\phi,T).x = p(\phi,T).\phi(T)$, and choose points v and w on this line as in Lemma 12.4, such that $T \subset \text{comv}(\{v,w\}) =: V$. By IIE, we obtain $\phi(V) = \phi(T)$.
If $p(\phi,S) = (0,1)$, then let u be the point of intersection of ℓ with the straight line $x_2 = \phi_2(S)$. (Cf. Fig. 12.4(a).) Then, by IIE applied to $S \subset \text{comv}(\{v,u\})$, we obtain $\phi(\text{comv}(\{v,u\})) = u$, but this contradicts Lemma 12.4.

Finally, if $p(\phi,S) > 0$, then let z be the point of intersection of ℓ with the straight line m with equation $p(\phi,S).x = p(\phi,S).\phi(S)$. (Cf. Fig. 12.4(b).) Note that $z_1 < \phi_1(V)$. Choose $y \in m$ as in Lemma 12.4, with y_2 so large that $S \subset \text{comv}(\{v,z,y\})$ (if necessary, we have to rechoose w in order to find an appropriate y). By IIE, $\phi(S) = \phi(\text{comv}(\{v,z,y\}))$, so by Lemma 12.4, $\phi(S) = \phi(V) = \phi(T)$.
So also for the case $p(\phi,T) > 0$, we have proved (12.1). ■

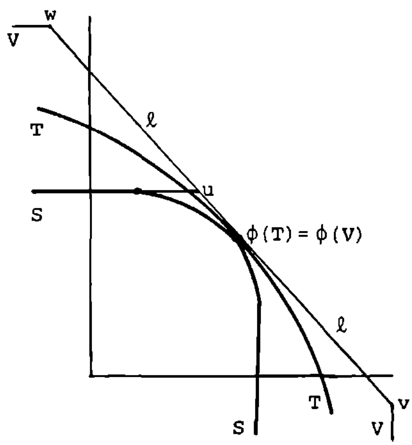


Figure 12.4(a) : $p(\phi,S) = (0,1)$

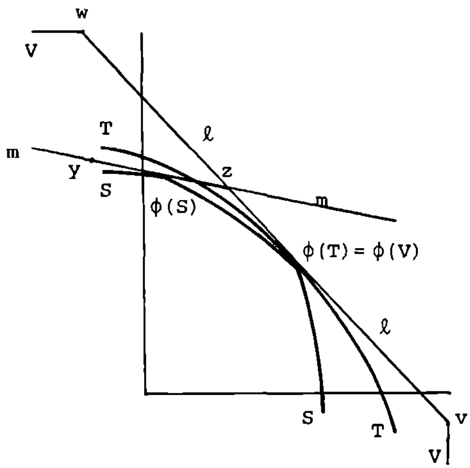


Figure 12.4(b) : $p(\phi,S) > 0$

Part B : Convention consistency

In this part, we show that the IIA-property in Theorem 11.8 (or the IIE-property in Theorem 12.5) can be replaced by the property of Def. 12.6, below. For $V,W \subset \mathbb{R}^2$, we denote by VW the set $\{vw : v \in V, w \in W\}$. For $S,T \in B$, we denote by $S * T$ the set $\text{com}((S \cap \mathbb{R}^2_+)(T \cap \mathbb{R}^2_+))$. Note that, if $S * T \in B$, then

$S * T \in B_+$.

Definition 12.6. A bargaining solution $\phi : B \to \mathbb{R}^2$ is called *convention consistent* if $\phi(S * T) = \phi(S)\phi(T)$ for all $S,T \in B$ with $S * T \in B_+$. (Convention consistency : CC)

At the end of the present section, we will give an interpretation of this property with the aid of the result of section 7. But first we prove the following theorem.

Theorem 12.7. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution. Then ϕ is Pareto optimal and convention consistent, if and only if $\phi \in N$.

The proof of this theorem is given after the following lemmas.

Lemma 12.8. Let the bargaining solution $\phi : B \to \mathbb{R}^2$ satisfy PO and CC. Then ϕ satisfies IR and STI, and $\phi(S) = \phi(S_+)$ for every $S \in B$.

Proof. Let $S \in B$ and $a \in \mathbb{R}^2_{++}$. Let $\square := \text{comv}(\{(1,1)\}) \in B_+$. Then, by CC and PO, $\phi(aS) = \phi(aS)\phi(\square) = \phi(aS * \square) = \phi(S * a\square) = a\phi(S)$. So STI holds, and further, by taking $a = (1,1)$, we obtain $\phi(S) = \phi(S * \square) = \phi(S_+)$: this also implies IR. ■

Lemma 12.9. Let $S \in B$ with $h(S) = (1,1)$. Then there exists a $T \in B$ with $h(T) = (1,1)$ and $S * T = \text{comv}(\{(1,0),(0,1)\})$.

Proof. Denote by $\Delta \in B_+$ the game $\text{comv}(\{(1,0),(0,1)\})$. Let $V := \{y \in \mathbb{R}^2_+ : x.y \leq 1 \text{ for all } x \in S,\ x \geq 0\}$. Then V is convex, V is bounded (for every $y \in V$, $(1,0).y \leq 1$ and $(0,1).y \leq 1$ imply $y \leq (1,1)$), and closed ($V = \bigcap_{x \in S, x \geq 0} \{y \in \mathbb{R}^2_+ : x.y \leq 1\}$). Further : $(1,0),(0,1) \in V$, and $(\alpha,0),(0,\alpha) \notin V$ if $\alpha > 1$, so $T := \text{com}(V) \in B_+$ with $h(T) = (1,1)$. By definition of V, we have $S * T \subset \Delta$. We still have to show : $\Delta \subset S * T$. Since $(1,0),(0,1) \in S \cap T$, we have $(1,0),(0,1) \in S * T$. Let $(t,1-t) \in P(\Delta)$ with $0 < t < 1$. Then $(t,1-t) = (z_1,z_2)(\frac{t}{z_1},\frac{1-t}{z_2}) \in S * T$ where $z = N^t(S)$ and $(\frac{t}{z_1},\frac{1-t}{z_2}) \in T$ in view of Lemma 12.1. So $\Delta \subset S * T$. ■

Proof of Theorem 12.7. If $\phi \in N$, then one immediately verifies that, besides PO, ϕ satisfies CC. Now suppose ϕ satisfies PO and CC. Let $S \in B$. Because

ϕ satisfies STI (Lemma 12.8), we suppose w.l.o.g. that $h(S) = (1,1)$. Then let $T \in B$ be as in Lemma 12.9, and $\Delta := \text{comv}(\{(1,0),(0,1)\})$. If $\phi(\Delta) = (1,0)$ then by CC, $\phi_1(S)\phi_1(T) = 1$, so $\phi_1(S) = 1$ and hence $\phi(S) = N^1(S)$. Similarly, $\phi(\Delta) = (0,1)$ implies $\phi(S) = N^0(S)$. Let now $\phi(\Delta) = (t,1-t)$ with $0 < t < 1$. If $x \in S$ and $x \neq N^t(S)$, then

$x_1^t x_2^{1-t} \phi_1(T)^t \phi_2(T)^{1-t} < (N_1^t(S)N_1^t(T))^t (N_2^t(S)N_2^t(T))^{1-t}$, hence

$x\phi(T) \neq (t,1-t) = N^t(\Delta) = \phi(\Delta) = \phi(S)\phi(T)$ (by CC). We conclude that $\phi(S) = N^t(S)$. ■

Remark 12.10. A $\phi \in N$ is determined by $\phi(\Delta)$ where, as in the foregoing proofs, $\Delta := \text{comv}(\{(1,0),(0,1)\})$: if $\phi(\Delta) = (t,1-t)$ for $0 \leq t \leq 1$, then $\phi = N^t$.

The last part of this section is devoted to a utility-theoretic foundation for the convention consistency property. Let $\Gamma_1 = \langle A,\bar{a},u^1,u^2\rangle$ and $\Gamma_2 = \langle B,\bar{b},v^1,v^2\rangle$ be 2-person bargaining situations, i.e. elements of BS. Let $\Gamma = \langle A \times B, \bar{c},w^1,w^2\rangle$ be a 2-person bargaining situation, such that conditions (7.1) and (7.2) hold with the $\bar{a}$ and $\bar{b}$ of (7.2) indeed the $\bar{a}$ and $\bar{b}$ of Γ_1 and Γ_2, respectively, and with $\bar{c} = (\bar{a},\bar{b})$. So, in view of Theorem 7.1, we may assume : $w^1(a,b) = u^1(a)v^1(b)$ and $w^2(a,b) = u^2(a)v^2(b)$ for all $a \in A$ and $b \in B$.

Let ϕ be a bargaining solution. If, for every $\ell \in L(A \times B)$, there are $\ell_1 \in L(A)$ and $\ell_2 \in L(B)$ such that $(Ew^1(\ell),Ew^2(\ell)) = (Eu^1(\ell_1)Ev^1(\ell_2),Eu^2(\ell_1)Ev^2(\ell_2))$, then the CC-property requires that the same equality holds for $\ell \in \text{alt}(\phi,\Gamma)$, $\ell_1 \in \text{alt}(\phi,\Gamma_1)$, and $\ell_2 \in \text{alt}(\phi,\Gamma_2)$. Suppose now, that ℓ_1 results in $a^k \in A$ with probability p_k, ℓ_2 in $b^j \in B$ with probability q_j, then, for $i = 1,2$, $Eu^i(\ell_1)Ev^i(\ell_2)$ is the expected utility Ew^i of the lottery in $L(A \times B)$ which results with probability p_kq_j in (a^k,b^j). So, according to the CC-property, each player is indifferent between this lottery and the lottery ℓ; hence, the CC- property may be interpreted as requiring each player to be indifferent between either playing Γ_1 and Γ_2 (ending in (a^k,b^j) with probability p_kq_j) or playing Γ (ending in the lottery ℓ).

The expression "convention consistency" is used in Binmore (1982) : that is the reason why we also use it. Our presentation differs from the presentation in that paper, and our result (Theorem 12.7) is more general since it also includes the solutions N^0 and N^1.

In this section, we shall introduce a simple market model, where two buyers, each endowed with an amount of money, together buy a continuum of goods. Of course, these buyers have certain preferences for these goods. We shall describe all competitive equilibria of such a market model, and show that Nash solutions N^t with $0 < t < 1$ give rise to special kinds of equilibria; one of the results is an interpretation of the weight t of a Nash solution N^t in terms of money. Other possible merits and future research directions are discussed at the end of the section.

In the following, λ denotes the (restriction to [0,1] of the) Lebesgue measure, $\mathcal{M}$ denotes the σ-algebra of (Lebesgue-) measurable subsets of [0,1], and $\mathcal{M}_0 \subset \mathcal{M}$ denotes the collection of sets of measure 0.

Definition 13.1. A *simple market* is a triple $E = \langle u^1, u^2, b \rangle$ such that :

(13.1) u^1 is a continuous nonincreasing function : $[0,1] \to [0,\infty)$.

(13.2) u^2 is a continuous nondecreasing function : $[0,1] \to [0,\infty)$.

(13.3) There is a $t \in (0,1)$ such that $u^1(t) \neq 0$ and $u^2(t) \neq 0$.

(13.4) $b \in \mathbb{R}^2_{++}$, $b_1 + b_2 = 1$.

(Note that, with these properties, u^1 and u^2 are Lebesgue - as well as Riemann - integrable, with positive integrals in view of especially (13.3).)

The interpretation of a simple market $E = \langle u^1, u^2, b \rangle$ is as follows. There are two *buyers* or *players*, called 1 and 2. Each buyer $i \in \{1,2\}$ has an amount of money $b_i > 0$. The buyers may purchase the *continuum of goods* [0,1] (every $t \in [0,1]$ is called a *good*) which costs $1 = b_1 + b_2$, provided they can reach some agreement on the division of the goods in [0,1]. We call u^i the *marginal utility function of buyer* i; for $M \in \mathcal{M}$, the Lebesgue-integral $\int_M u^i d\lambda$ is well-defined. It is called the *utility of* M *for* i.

The monotonicity properties in (13.1) and (13.2) can be interpreted, roughly, as saying that buyer 1 [2] likes best the goods close to good 0 [1]. Condition (13.3) as well as continuity of the u^i, are required mainly for mathematical convenience.

Example 13.2. Let the buyers be two broadcasting companies who want to buy the time between seven (=0) and eight (=1) in the evening for broadcasting.

Their preferences are given by marginal utility functions u^i which satisfy (13.1) - (13.3); in particular, the monotonicity properties in (13.1) and (13.2) reflect the facts that broadcasting company nr. 1 aims especially at very young children, whereas the other broadcasting company's programme is unsuited for young children.
Further, we suppose that only the whole hour between seven and eight is for sale, and not parts of it; in this case, the two broadcasting companies may decide to buy the hour together - none of them is interested in buying the whole hour - provided they can reach some agreement on the division of the time, afterwards; the amounts of money b_i are fixed, may be for budgetary reasons.

A topic of future research may be the case where the investment decisions b_i depend on the corresponding divisions of the hour of time, given some division rule depending on the amounts b_i.

We let E denote the family of all simple markets. We propose the following solution concept for a simple market.

<u>Definition 13.3.</u> Let $E = \langle u^1, u^2, b\rangle \in E$. A *competitive equilibrium* (c.e.) for E is a triple $\langle M,N,\pi\rangle$ where :

(13.5) $M,N \in \mathcal{M}$ and $M \cap N \in \mathcal{M}_0$;

(13.6) π is a *price function*, i.e. a strictly positive continuous function on $[0,1]$ with $\int_{[0,1]} \pi d\lambda = 1$;

(13.7) $\int_M \pi d\lambda = b_1$, $\int_N \pi d\lambda = b_2$;

(13.8) $\int_M u^1 d\lambda \geq \int_{M'} u^1 d\lambda$ for all $M' \in \mathcal{M}$ with $\int_{M'} \pi d\lambda \leq b_1$;

(13.9) $\int_N u^2 d\lambda \geq \int_{N'} u^2 d\lambda$ for all $N' \in \mathcal{M}$ with $\int_{N'} \pi d\lambda \leq b_2$.

Note that, from the definition of a c.e. $\langle M,N,\pi\rangle$, it follows that $[0,1] \setminus (M \cup N)$ has measure 0.

In a c.e. $\langle M,N,\pi\rangle$, player 1[2] gets M[N], which is the best he can get for his money $b_1[b_2]$ at prices given by π; of course, we call $\int_M \pi d\lambda$ the *price of* M, for $M \in \mathcal{M}$.

We shall describe all c.e.'s of a simple market E, and single out special ones with the aid of bargaining solutions. For that reason, we shall associate bargaining games with simple markets. However, we first give some preparatory definitions and lemmas.

Definition 13.4. By $\mathcal{A}$, we denote the family of all *allocation problems*, i.e. pairs $A = \langle u^1,u^2\rangle$ where u^1 and u^2 satisfy (13.1) - (13.3). For an $E = \langle u^1,u^2,b\rangle \in \mathcal{E}$, we call $A(E) := \langle u^1,u^2\rangle \in \mathcal{A}$ the *allocation problem corresponding to* E. We denote by $U^1(t) := \int_{[0,t]} u^1 d\lambda$ the utility of $[0,t]$ for player 1, and by $U^2(t) := \int_{[t,1]} u^2 d\lambda$ the utility of $[t,1]$ for player 2.

Lemma 13.5. Let $A \in \mathcal{A}$, $A = \langle u^1,u^2\rangle$. Then U^1 and U^2 are continuous and concave functions on $[0,1]$. Further, U^1 is nondecreasing and U^2 is nonincreasing, $U^1(0) = U^2(1) = 0$. Finally, U^1 and U^2 are differentiable on $(0,1)$ with derivatives u^1 and $-u^2$, respectively.

Proof. Elementary. ∎

Lemma 13.6. Let $M,N \in \mathcal{M}$ with $M \cap N \in \mathcal{M}_0$, and let $\langle u^1,u^2\rangle \in \mathcal{A}$. Then a $t \in [0,1]$ exists with $U^1(t) \geq \int_M u^1 d\lambda$ and $U^2(t) \geq \int_N u^2 d\lambda$.

Proof. Since U^1 is continuous (Lemma 13.5), there exists a $t \in [0,1]$ with $U^1(t) = \int_M u^1 d\lambda$, so $\int_M u^1 d\lambda = \int_{[0,t]\setminus M} u^1 d\lambda + \int_{[0,t]\cap M} u^1 d\lambda$. From this it follows that $\int_{M\cap[t,1]} u^1 d\lambda = \int_{[0,t]\setminus M} u^1 d\lambda$. Hence, in view of (13.1), we obtain $\lambda([0,t]\setminus M) \leq \lambda(M\cap[t,1])$, so $\lambda(M) \geq \lambda([0,t]) = t$. From this, we have $\lambda(N) \leq 1-t$, and hence, by a reversed argument using (13.2), we obtain $\int_N u^2 d\lambda \leq U^2(t)$. ∎

Lemma 13.7. Let $E = \langle u^1,u^2,b\rangle \in \mathcal{E}$ and let $\langle M,N,\pi\rangle$ and $\langle M',N',\pi\rangle$ be c.e.'s for E. Then :

(i) $\int_M u^1 d\lambda = \int_{M'} u^1 d\lambda$, $\int_N u^2 d\lambda = \int_{N'} u^2 d\lambda$.

(ii) There exists a $t \in [0,1]$ such that also $\langle [0,t],[t,1],\pi\rangle$ is a c.e.

Proof. (i) Straightforward from the definition of a c.e. (ii). Take $t \in [0,1]$ such that $\int_{[0,t]} \pi d\lambda = b_1$, $\int_{[t,1]} \pi d\lambda = b_2$ (cf. (13.6)). By similar arguments as used in the proof of Lemma 13.6, we derive from $\int_{[0,t]} u^1 d\lambda \le \int_M u^1 d\lambda$ and $\int_{[t,1]} u^2 d\lambda \le \int_N u^2 d\lambda$, the facts $\lambda(M) \geqq t$ and $\lambda(N) \geqq 1-t$, so $\lambda(M) = t$, $\lambda(N) = 1-t$, $\int_{[0,t]} u^1 d\lambda = \int_M u^1 d\lambda$, and $\int_{[t,1]} u^2 d\lambda = \int_N u^2 d\lambda$.

Hence $<[0,t],[t,1],\pi>$ is a c.e. ∎

Lemma 13.6 expresses the fact that, in an allocation problem, allocations of the form $[0,t]$ for player 1 and $[t,1]$ for player 2, are "Pareto optimal". Lemma 13.7 expresses the strongly related fact that, in a simple market, every c.e. $<M,N,\pi>$ is "utility-equivalent" to a c.e. of the form $<[0,t],[t,1],\pi>$.

<u>Lemma 13.8.</u> Let $A = <u^1,u^2> \in \mathcal{A}$. Let $\underline{s} := 0 \vee \sup\{t \in [0,1] : u^2(t) = 0\}$ and $\bar{s} := 1 \wedge \inf\{t \in [0,1] : u^1(t) = 0\}$.
Then :

(i) $0 \leqq \underline{s} \leqq \bar{s} \leqq 1$,

(ii) $S(A) := \text{com}(\{(\alpha, U^2 \circ (U^1)^{-1}(\alpha) : U^1(\underline{s}) \le \alpha \le U^1(\bar{s})\}) \in B$,

(iii) $P(S(A)) = \{(\alpha, U^2 \circ (U^1)^{-1}(\alpha)) : U^1(\underline{s}) \leqq \alpha \leqq U^1(\bar{s})\}$.

<u>Proof.</u> (i) Follows, in particular, from (13.3). (ii) $U^2 \circ (U^1)^{-1} : [U^1(\underline{s}), U^1(\bar{s})] \to [0,\infty)$ is concave and strictly decreasing, by Lemma 13.5 and the choice of $\underline{s}$ and $\bar{s}$. In view of (13.3), $x > 0$ for some $x \in S(A)$. We conclude that $S(A) \in B$. (iii) See the proof of (ii). ∎

In view of Lemma 13.8, we may call, for each allocation problem A, the game S(A) the *bargaining game corresponding to* A. Also, for $E \in \mathcal{E}$ with $A(E) = A$, we call $S(E) := S(A)$ the *bargaining game corresponding to* E.

For an $A \in \mathcal{A}$, the corresponding bargaining game S(A) has the property that for every $x \in P(S(A))$ with $x > 0$ there is a unique supporting line of S(A) at x, i.e. S(A) is smooth at every $x \in P(S(A))$, $x > 0$ (cf. section 12, after Def. 12.2). This is a consequence of the last statement in Lemma 13.5. Further, $S(A) \in B_+$ by definition. Let $B_+^s \subset B_+$ denote the subfamily of B_+ consisting of all bargaining games with the mentioned property ("s" from "smooth"). Then not only we have $S(A) \in B_+^s$ for every $A \in \mathcal{A}$, but also :

Lemma 13.9. Let $S \in B_+^s$. Then there exists an $A \in \mathcal{A}$ with $S(A) = S$.

Proof. For $t \in [0,1]$, let $(U^1(t),U^2(t))$ be the point of intersection of $W(S)$ and the straight line through the point $h(S)(t,1-t)$ with slope $h_2(S)h_1(S)^{-1}$. In this way, the functions $U^1,U^2 : [0,1] \to [0,\infty)$ are determined. Let now, on $(0,1)$, $u^1 := (U^1)'$ and $u^2 := -(U^2)'$, and let for $i = 1,2$, $u^i : [0,1] \to [0,\infty)$ be the continuous extension of $u^i : (0,1) \to [0,\infty)$. We leave it to the reader to verify that $A := \langle u^1,u^2\rangle \in \mathcal{A}$ and $S(A) = S$. ■

The now following theorem gives a characterization of all competitive equilibria of the form $\langle[0,t],[t,1],\pi\rangle$ in a simple market. Cf. Lemma 13.7.

Theorem 13.10. Let $E = \langle u^1,u^2,b\rangle \in \mathcal{E}$ and $t \in [0,1]$. Then the following two statements are equivalent.

(a) $\langle[0,t],[t,1],\pi\rangle$ is a c.e. of E.

(b) (i) For every $s \in [0,t]$, we have $\frac{u^1(s)}{\pi(s)} \geq \frac{u^1(t)}{\pi(t)}$, and for every $s \in [t,1]$, we have $\frac{u^1(s)}{\pi(s)} \leq \frac{u^1(t)}{\pi(t)}$.

(ii) For every $s \in [0,t]$, we have $\frac{u^2(s)}{\pi(s)} \leq \frac{u^2(t)}{\pi(t)}$, and for every $s \in [t,1]$, we have $\frac{u^2(s)}{\pi(s)} \geq \frac{u^2(t)}{\pi(t)}$.

(iii) π is a price function with $\int_{[0,t]} \pi d\lambda = b_1$ (and $\int_{[t,1]} \pi d\lambda = b_2$).

Proof. We first show the implication (b) $\Rightarrow$ (a). Assume (b) and let $M \in \mathcal{M}$ with $\int_M \pi d\lambda \leq b_1$. We want to show that $\int_M u^1 d\lambda \leq \int_{[0,t]} u^1 d\lambda$.

This is obviously true, in view of (13.1), if $u^1(t) = 0$. Otherwise we have, with the aid of (b) (i):

$$\int_{M\cap[t,1]} u^1 d\lambda \leq \frac{u^1(t)}{\pi(t)} \int_{M\cap[t,1]} \pi d\lambda \leq \frac{u^1(t)}{\pi(t)} (b_1 - \int_{M\cap[0,t]} \pi d\lambda) = \frac{u^1(t)}{\pi(t)} \int_{[0,t]\setminus M} \pi d\lambda$$

$\leq \int_{[0,t]\setminus M} u^1 d\lambda$. Hence $\int_M u^1 d\lambda \leq \int_{[0,t]} u^1 d\lambda$. Similarly, one shows that

$\int_N u^2 d\lambda \leq \int_{[t,1]} u^2 d\lambda$ for any $N \in \mathcal{M}$ with $\int_N \pi d\lambda \leq b_2$. We conlude that (a) holds.

Now the implication (a) $\to$ (b). Assume (a), then we still have to show (i) and (ii) of (b). We only show (i). Since (i) is obviously true if $u^1(t) = 0$, we suppose $u^1(t) \neq 0$. As to the first inequality in (i), suppose to the contrary that $\pi(\bar{s}) = (1 + \varepsilon) \frac{\pi(t)}{u^1(t)} u^1(\bar{s})$ for some $\bar{s} \in [0,t)$ (note that $t \neq 0,1$ by (a), in particular since b > 0) and $\varepsilon > 0$. Since π and u^1 are continuous, we can find $\gamma, \delta > 0$ small enough to give $\int_{[\bar{s},\bar{s}+\gamma]} \pi d\lambda = \int_{[t,t+\delta]} \pi d\lambda$ and

$\pi(s) > (1 + \frac{\varepsilon}{2}) \frac{\pi(t)}{u^1(t)} u^1(s)$ for all s with $\bar{s} \leqq s \leqq \bar{s}+\gamma$ whereas

$\pi(s) < (1 + \frac{\varepsilon}{2}) \frac{\pi(t)}{u^1(t)} u^1(s)$ for all s with $t \leqq s \leqq t+\delta$. Hence

$\int_{[\bar{s},\bar{s}+\gamma]} u^1(s)ds < \int_{[t,t+\delta]} u^1(s)ds$ which means that buyer a can improve his allocation without violating his budget restriction, in contradiction with (a). We have proved the first inequality of (i). Secondly and similarly, if the second inequality in (i) would not hold, then bargainer 1 could improve his allocation by bying, instead of $(t-\zeta,t]$ for a small $\zeta > 0$, an equally expensive interval $[s',s'+\eta)$ allocated to bargainer 2 in the given c.e. ∎

Theorem 13.10 shows what a c.e. $\langle[0,t],[t,1],\pi\rangle$ for a simple market $\langle u^1,u^2,b\rangle$ looks like (cf. Fig. 13.1) : for $i = 1,2$, if $u^i(t) \neq 0$, then multiply u^i by $\alpha^i = \frac{\pi(t)}{u^i(t)} > 0$, so $\alpha^i u^i(t) = \pi(t)$; then, if $u^1(t) \neq 0$, we have $\pi(s) \leqq \alpha^1 u^1(s)$ for all $s \in [0,t]$, whereas $\pi(s) \geq \alpha^1 u^1(s)$ for all $s \in [t,1]$; and, if $u^2(t) \neq 0$, then we have $\pi(s) \geqq \alpha^2 u^2(s)$ for all $s \in [0,t]$ whereas $\pi(s) \leqq \alpha^2 u^2(s)$ for all $s \in [t,1]$. So, if both $u^1(t)$ and $u^2(t)$ are positive, then the graph of π lies in between the graphs of $\alpha^1 u^1$ and $\alpha^2 u^2$ everywhere.

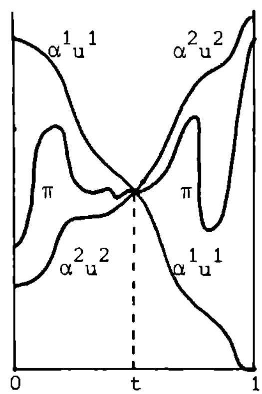


Figure 13.1.

Corollary 13.11. Let $E = \langle u^1, u^2, b\rangle$ be a simple market. Then :

(i) There exists a c.e. $\langle[0,t],[t,1],\pi\rangle$ for E.

(ii) If $t \in [0,1]$ is such that $(U^1(t),U^2(t)) \geq b(U^1(1),U^2(0))$, then there exists a price function π such that $\langle[0,t]\ [t,1],\pi\rangle$ is a c.e. for E.

(iii) If $\langle[0,t],[t,1],\pi\rangle$ is a c.e. for E, then $(U^1(t),U^2(t)) \geqq b(U^1(1),U^2(0))$.

Proof. (i) follows (ii) since a point t as in (ii) always exists : take $t = b_1$, then $U^1(t) \geqq b_1U^1(1)$ since u^1 is nonincreasing and $U^2(t) \geqq b_2U^2(0)$ since u^2 is nondecreasing. Now let t be as in (ii). If $u^1(t) = 0$, then $u^2(t) \neq 0$ in view of (13.3) and the monotonicity properties of u^1 and u^2; then take a strictly positive continuous function $p : [0,1] \to \mathbb{R}$ such that $p(t) = u^2(t)$, $\int_{[t,1]} p d\lambda \left(\int_{[0,1]} p d\lambda\right)^{-1} = b_2$, and $p(s) \geqq u^2(s)$ for all $0 \leqq s \leqq t$ whereas $p(s) \leqq u^2(s)$ for all $t \leqq s \leqq 1$. Take $\pi(s) := p(s)\left(\int_{[0,1]} p d\lambda\right)^{-1}$ for every $s \in [0,1]$, then it follows from Theorem 13.10 that $\langle[0,t],[t,1],\pi\rangle$ is a c.e. Similarly if $u^2(t) = 0$. If $u^1(t) \neq 0$ and $u^2(t) \neq 0$, then let $\alpha > 0$ be such that $\alpha u^1(t) = u^2(t)$, and take p and π as above, such that, additionally, the graph of p lies in between the graphs of αu^1 and u^2 everywhere. Then, again by Theorem 13.10, $\langle[0,t],[t,1],\pi\rangle$ is a c.e. We have shown ((i) and) (ii). Statement (iii) is straightforward from Theorem 13.10. ∎

Before studying the relations between bargaining solutions and competitive equilibria, we need a few additional definitions.

Definition 13.12. Let $b \in \mathbb{R}^2_+$ with $b_1 + b_2 = 1$, and let $\phi : B \to \mathbb{R}^2$ be a bargaining solution. We call ϕ *b-monotonic* if $\phi(S) \geqq bh(S)$ for every $S \in B$. We call ϕ *symmetrically monotonic* if $\phi(S) \geqq \frac{1}{2}h(S)$ for every $S \in B$ (cf. Sobel (1981)).

We shall use the b-monotonicity property not only in this section : it will reappear in section 27.

Definition 13.13. Let $E = \langle u^1, u^2, b\rangle$. If $M,N \in \mathcal{M}$, we call the pair $\langle M,N\rangle$ an *equilibrium allocation for* E if $\langle M,N,\pi\rangle$ is a c.e. for E, for some price function π. Further, we call $(\alpha,\beta) \in \mathbb{R}^2$ a *pair of equilibrium utilities for* E if an equilibrium allocation $\langle M,N\rangle$ for E exists with $\int_M u^1 d\lambda = \alpha$ and

$\int_N u^2 d\lambda = \beta$.

The following theorem is an immediate consequence of Lemmas 13.6 and 13.9, and Corollary 13.11 (ii), (iii).

Theorem 13.14. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution, and let $b \in \mathbb{R}^2_{++}$ with $b_1 + b_2 = 1$. The following two assertions are equivalent.

(i) For every $S \in B^s_+$, $\phi(S)$ is an equilibrium utility allocation for every $E = \langle .,.,b\rangle \in \mathcal{E}$ such that $S = S(E)$.

(ii) ϕ is b-monotonic and weakly Pareto optimal on B^s_+.

Remark 13.15. In section 27 we will see that many bargaining solutions appearing in this monograph, satisfy b-monotonicity for some b. So Theorem 13.14 often applies. Theorem 13.14 provides an interpretation of the parameter b in the b-monotonicity property in terms of money. Similarly, the now following theorem gives an interpretation of the parameter q of a non-symmetric Nash solution $N^q (0 < q < 1)$. This has been the motivation for us to give the present section the place it has in this monograph.

Theorem 13.16. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution, and let $0 < q < 1$. The following two assertions are equivalent.

(i) $\phi(S) = N^q(S)$ for every $S \in B^s_+$.

(ii) For every $S \in B^s_+$ and $E = \langle u^1, u^2, (q,1-q)\rangle \in \mathcal{E}$ with $S(E) = S$, $\phi(S)$ is the pair of equilibrium utilities corresponding to the c.e. $\langle [0,t],[t,1],\pi\rangle$ where π is a price function such that $\pi(s) = \frac{\pi(t)}{u^1(t)} u^1(s)$ if $s \in [0,t]$ and $\pi(s) = \frac{\pi(t)}{u^2(t)} u^2(s)$ if $s \in [t,1]$.

Proof. (i) $\Rightarrow$ (ii). Assume (i), and let $S \in B^s_+$ and $E = \langle u^1, u^2, (q,1-q)\rangle \in \mathcal{E}$ with $S(E) = S$. Then $\phi(S) = N^q(S) = (U^1(t), U^2(t))$ where $t \in [0,1]$ maximizes the expression $U^1(s)^q U^2(s)^{1-q}$. A necessary and sufficient condition for this is $\frac{d}{dt} U^1(t)^q U^2(t)^{1-q} = 0$, or equivalently :

$$(13.10) \qquad \frac{q}{1-q} = \frac{U^1(t)u^2(t)}{U^2(t)u^1(t)} .$$

We have to show that $\langle [0,t],[t,1],\pi\rangle$ with π as in the theorem, is a c.e.; in view of Theorem 13.10, it is sufficient to show that

$$\frac{\int_{[0,t]} \pi d\lambda}{\int_{[t,1]} \pi d\lambda} = \frac{q}{1-q},$$

and this follows straightforwardly from (13.10).

The implication (ii) $\Rightarrow$ (i) follows by reversing the above argument. ■

Thus, for an equilibrium allocation determined by a nonsymmetric Nash solution N^q with $0 < q < 1$ there is a price function proportional for each one of the bargainers to the marginal utilities of the goods belonging to that bargainer's share. Two other special cases are summarized in the following theorem, a proof of which can be given with the aid of Theorem 13.10 again, and is omitted.

Theorem 13.17. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution, and let $b \in \mathbb{R}^2_{++}$ with $b_1 + b_2 = 1$. The following two assertions are equivalent :

(i) ϕ is weakly Pareto optimal on B^s_+, with $\phi_1(S) = b_1h_1(S)$ [$\phi_2(S) = b_2h_2(S)$, respectively] for every $S \in B^s_+$.

(ii) For every $S \in B^s_+$ and $E = \langle u^1,u^2,b\rangle \in E$ with $S(E) = S$, $\phi(S)$ is the pair of equilibrium utilities corresponding to the c.e. $\langle[0,t],[t,1],\pi\rangle$ where π is a price function such that $\pi(s) = \frac{\pi(t)}{u^1(t)} u^1(s)$ [$\pi(s) = \frac{\pi(t)}{u^2(t)} u^2(s)$, resp.] for all $s \in [0,1]$.

We conclude with some discussion. The concept of a simple market may be generalized into several directions : higher dimension of the continuum of goods, more than two buyers; or relaxation, within the present model, of conditions (13.1) and (13.2). As to the last point : it is not difficult to see that Theorem 13.10 does not hold anymore if the continuity requirements in (13.1) or (13.2) are dropped. (E.g. let $u^1(s) = 2$ if $0 \leqq s < \frac{3}{8}$, $u^1(s) = \frac{3}{2} - s$ if $\frac{3}{8} \leqq s \leqq \frac{1}{2}$, $u^1(s) = 1$ if $\frac{1}{2} \leq s \leqq 1$, $u^2(s) = \frac{9}{8}$ for all $0 \leqq s \leqq 1$, $\pi(s) = \frac{9}{8}$ if $0 \leq s \leqq \frac{3}{8}$, $\pi(s) = \frac{3}{2} - s$ if $\frac{3}{8} \leqq s \leqq \frac{5}{8}$, $\pi(s) = \frac{7}{8}$ if $\frac{5}{8} \leq s \leqq 1$; and let $b_1 = \frac{27}{64}$, $b_2 = \frac{37}{64}$. Then $E = \langle u^1,u^2,b\rangle$ satisfies all conditions for a simple market except continuity of u^1, $\langle[0,\frac{3}{8}],[\frac{3}{8},1],\pi\rangle$ satisfies the conditions for a c.e.; but (b) (i) of Theorem 13.10 does not hold.) Perhaps, dropping the continuity requirements in (13.1) and (13.2) must be matched by dropping the continuity condition for a price function π.

The monotonicity conditions in (13.1) and (13.2) are quite restrictive : we conjecture, however, that these can be dropped without excluding a result in the same spirit as Theorem 13.10.

Another line of research is to seek for possible relations between competitive equilibria in a simple market, and other bargaining solutions than the ones occuring in Theorems 13.16 and 13.17.

Our final points are made in the following two remarks.

<u>Remark 13.18.</u> Let $A = \langle u^1, u^2 \rangle \in \mathcal{A}$, which means that u^1 and u^2 satisfy conditions (13.1) - (13.3). Without altering the definition of such an allocation problem, we may give it a different interpretation, as follows. There is one unit of a perfectly divisible good (e.g., one litre of wine). And a "division" $t \in [0,1]$ means : buyer 1 gets t, whereas buyer 2 gets $1-t$. With this interpretation, only the integrals $\int_{[0,t]} u^1 d\lambda = U^1(t)$ and $\int_{[t,1]} u^2 d\lambda = U^2(t)$ are relevant, and interpreted as the utilities of $100\,t$ percent and $100(1-t)$ percent of the good for buyers 1 and 2, respectively.

<u>Remark 13.19.</u> In (ii) of Theorem 13.14, we may replace B_+^s by B_+ if ϕ is continuous or Pareto-continuous : the latter properties are introduced in section 19.

First versions of the results in this section were given in Peters (1985 a).

14. OTHER MODELS FOR NASH SOLUTIONS

Let $S \in B$, $n_1, n_2 \in \mathbb{N}$, and consider the following 2-*person game in normal form*, which we can associate with the triple S, n_1, and n_2. Each player $i \in \{1,2\}$ has the strategy space $P(S) \cap \mathbb{R}^2_{++}$. If player 1 plays x and player 2 plays y, then the payoff to player 1 is given by :

$$K_1(x,y) := \begin{cases} x_1 & \text{if } x_1 \leq y_1 \\ x_1 & \text{if } x_1 > y_1 \text{ and } R(x) \geq R(y) \\ y_1 & \text{if } x_1 > y_1 \text{ and } R(x) < R(y). \end{cases}$$

Here, $R(Z) := z_1^{n_1(n_1+n_2)^{-1}} z_2^{n_2(n_1+n_2)^{-1}}$. Similarly, the payoff to player 2 is given by :

$$K_2(x,y) := \begin{cases} y_2 \text{ if } y_2 \leq x_2 \\ y_2 \text{ if } y_2 > x_2 \text{ and } R(y) > R(x) \\ x_2 \text{ if } y_2 > x_2 \text{ and } R(y) \leqq R(x). \end{cases}$$

A *Nash equilibrium (in pure strategies)* for such a game is a pair of strategies $(\bar{x},\bar{y})$ satisfying $K_1(\bar{x},\bar{y}) \geqq K_1(x,\bar{y})$ for every strategy x and $K_2(\bar{x},\bar{y}) \geqq K_2(\bar{x},y)$ for every strategy y. The reader will probably not be surprised by the following lemma.

Lemma 14.1. In the game described above, the unique pair of utilities corresponding to a Nash equilibrium, is
$N^{n_1(n_1+n_2)^{-1}}(S)$.

Proof. Straightforward. ■

The above model is a "noncooperative implementation" of the nonsymmetric Nash-solution $N^{n_1(n_1+n_2)^{-1}}$. One may think of the players' strategies as of proposals in a round of actual bargaining.

In the model as it is, the parameters n_1 and n_2, as well as the function R, are left unaccounted for. However, we shall remedy this defect below : this will occupy the larger part of the present section, which we will conclude by mentioning some other models in literature that are somehow related to Nash solutions.

Zeuthen (1930), avant la lettre, derived the (symmetric) Nash solution by means of describing a negotiation process. This fact was first recognized by Harsanyi (1956). Kalai (1977a) derived nonsymmetric Nash solutions by considering certain replications of 2-person bargaining games. We now present a model which also leads to nonsymmetric Nash solutions, and which is a mixture of the ideas of Harsanyi - Zeuthen and Kalai.

We describe this model without attempting to attain a rigourous formalism. We want to discover a set of assumptions under which a bargaining process may take the form of the normal-form game described at the beginning of this section.

Let $S \in B$ be a 2-person bargaining game, and let $0 < t < 1$. Let n_1 and n_2 be positive integers, and let $t = n_1(n_1 + n_2)^{-1}$. Our model will lead to the nonsymmetric Nash solution N^t(, where t is a rational number, of course; we note that $N^t(S)$ for t irrational can be approximated by $N^q(S)$ with q rational). We suppose that there is a family 1 of n_1 players of type 1 and a family 2 of n_2 players of type 2, and that each family chooses one of its members as its representative, say P_1 and P_2 respectively, who play the bargaining game S on behalf of their respective families. We assume that all the players of one type (i.e., in one family) are "replicas" of each other, which concept we take to be defined by assumptions (14.1) - (14.3) below. These assumptions are intended to justify the calculations to be made below.

(14.1) If decisions are to be made, all the members of a family decide indepedently of each other.

(14.2) Any decision of a family to comply to some proposal of the opposite family, must be unanimous.

(14.3) That family, which is less willing to risk a conflict, will be the first to comply to some proposal of the opposite family. As a measure of "willingness to risk a conflict", we take the maximum value, for which it is still worthwhile to stick to one's proposal, of the subjective probability that also the representative of the opposing family will stick to his proposal.

Assumption (14.3) which is analogous to Zeuthen's central assumption, certainly has some intuitive appeal. See also Harsanyi (1956), or Roth (1979).

We proceed with the actual description of the game. There is only one round, in which the representative P_i of each family i makes a proposal. A proposal is an element of $P(S) \cap \mathbb{R}^2_{++}$. Suppose $x = (x_1, x_2)$ is the proposal of P_1, meaning that he proposes family i to obtain utility-payoff x_i $(i=1,2)$. Suppose, similarly, that $y = (y_1, y_2)$ is the proposal of P_2. Then (by some legal institution) an outcome in $\{0, x, y, (x_1, y_2), (y_1, x_2)\}$ is enforced as the final outcome of the game, as follows. The final outcome is: (x_1, y_2) if $x_1 \leqq y_1$; (y_1, x_2) if $x_1 > y_1$ and both representatives are willing to concede to each other's proposals; x if $x_1 > y_1$ and P_2 concedes to P_1's proposal, whereas P_1 does not concede to P_2's proposal; y if $x_1 > y_1$ and P_1 concedes to P_2's proposal, whereas P_2 does not concede to P_1's proposal; and 0 if $x_1 > y_1$ and no one is willing to concede.

Now suppose P_1(and, in fact, every member of family 1) assesses a subjective probability q that P_2 (and thus, any member of family 2) will <u>not</u>

concede to proposal x, where we presume that $x_1 > y_1$, that is, x and y are incompatible. According to the assumptions (14.1) and (14.2), P_1 calculates the expected utility for family 1 of <u>not</u> conceding to y as

$$(14.4)\quad E^1(x,y) = (1-q)^{n_2} x_1 + (1-(1-q)^{n_2}).0$$

Not conceding to y is strictly better for family 1 than conceding if and only if

$$(14.5)\quad E^1(x,y) > y_1 .$$

Combining (14.4) and (14.5), we find that not conceding to y is strictly better for family 1 than conceding if and only if

$$(14.6)\quad q < 1 - (y_1 x_1^{-1})^{n_2^{-1}}$$

Similarly if we let r be the subjective probability, as assessed by P_2, that P_1 (and thus, any member of family 1) will <u>not</u> concede to proposal y, we find that <u>not</u> conceding to x is strictly better for family 2 if and only if

$$(14.7)\quad r < 1 - (x_2 y_2^{-1})^{n_1^{-1}} .$$

According to assumption (14.3), we conclude from (14.6) and (14.7) that P_1 or, equivalently, family 1 will concede to proposal y if and only if

$$1 - (y_1 x_1^{-1})^{n_2^{-1}} < 1 - (x_2 y_2^{-1})^{n_1^{-1}} \text{, i.e. if and only if}$$

$$(14.8)\quad x_1^t x_2^{1-t} < y_1^t y_2^{1-t} .$$

In other words, family 1 concedes to y if and only if in x the "Nash product with weight t" has a lower value than in y. If we adopt the convention that family 2 concedes to x if $x_1^t x_2^{1-t} = y_1^t y_2^{1-t}$, then the described bargaining process reduces to the normal-form game of Lemma 14.1.

Within the limits set by assumptions (14.1) - (14.3), this model gives - again, see section 13, especially Remark 13.15 and Theorem 13.16 - an interpretation of the weight t of a nonsymmetric Nash solution N^t $(0 < t < 1)$.

<u>Remark 14.2.</u> Some other references in which the Nash solution or nonsymmetric Nash solutions play important or central roles, are : Anbar and Kalai (1978), Aumann and Kurz (1977), Roth (1978), van Damme (1984), Nakayama (1983). Still another characterization of the family $\mathcal{N}$ will be given in section 20.

MONOTONICITY PROPERTIES

In a paper on arbitration games, Raiffa (1953) has proposed some (2-person) arbitration schemes or bargaining solutions. One of these is Nash's solution, and another one will be studied in this chapter, section 15. The latter solution was characterized axiomatically by Kalai and Smorodinsky (1975), who replaced, in Nash's result, the IIA-property by a property called "individual monotonicity" here. We shall extend this result in section 15. In section 16, we shall characterize a family of 2-person bargaining solutions satisfying a related monotonicity property called "global individual monotonicity". Thereby we extend a result of Kalai and Rosenthal (1978). Several other monotonicity properties have been proposed in literature : see, e.g. Thomson and Myerson (1980). See also Rosenthal (1976), for a solution, closely related to the above mentioned bargaining solution of Raiffa (or Kalai and Smorodinsky).

15. INDIVIDUAL MONOTONICITY

In a discussion of the Nash bargaining model, Kalai and Smorodinsky (1975) raised an objection against the independence of irrelevant alternatives property. To illustrate their objection, they gave the following example.

Example 15.1. (Cf. Fig. 15.1.) Let $S := \text{comv}(\{(1,0),(0,1),(\frac{3}{4},\frac{3}{4})\})$ and $T := \text{comv}(\{(1,0),(0,1),(1,\frac{7}{10})\})$. Then, for any $-\infty < \lambda \le 1$, $f_S^2(\lambda) \le f_T^2(\lambda)$ (cf. Def. 9.12). Yet $N(S) = (\frac{3}{4},\frac{3}{4})$ and $N(T) = (1,\frac{7}{10})$, so player 2 gets less in T than in S.

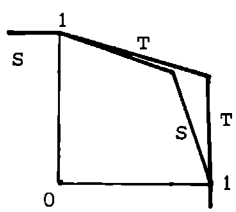


Figure 15.1.

Kalai and Smorodinsky gave the following alternative for the IIA-property. They called their property "Axiom of Monotonicity". In this property, the utopia point $h(S)$ of a game $S \in B$ plays a crucial role (see Def. 9.10).

Definition 15.2. A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ is called *individually monotonic* if, for all $i,j \in \{1,2\}$ with $i \neq j$ and $S,T \in B$ with $S \subset T$ and $h_i(S) = h_i(T)$, we have $\phi_j(S) \leq \phi_j(T)$.
(Individual monotonicity : IM)

The following 2-person bargaining solution was proposed by Raiffa (1953).

Definition 15.3. To every $S \in B$, the 2-person bargaining solution KS assigns the (unique) point of intersection of $P(S)$ with the straight line through 0 and $h(S)$. We call KS the *Kalai - Smorodinsky solution*.

We call this solution the Kalai - Smorodinsky solution in view of the following theorem, which was derived in Kalai and Smorodinsky (1975) - in a slightly weaker form : they used PO and AN instead of WPO and SYM, respectively -, and which differs from Theorem 11.4 in that IIA is replaced by IM. Its proof is readily derived from Kalai and Smorodinsky's proof.

Theorem 15.4. The solution $KS : B \to \mathbb{R}^2$ is the unique 2-person bargaining solution with the properties WPO, STI, SYM, and IM.

The main purpose of this section is an extension of Theorem 15.4, to be obtained by the dropping of SYM. On the other hand, we strengthen WPO to PO. See the following example, which shows that, in order to guarantee (strong) Pareto optimality, it is in this context not sufficient to require WPO.

Example 15.5. Let $\phi : B \to \mathbb{R}^2$ be the 2-person bargaining solution which assigns $(0,f_S^2(0))$ to every $S \in B$. Then ϕ satisfies IR,STI,IM, and WPO but not PO.

Remark 15.6. If we consider solutions defined on a class of "bargaining games" which are compact instead of comprehensive, then, in the presence of IM, it can be proved that WPO implies PO. See Peters and Tijs (1985a), on which this chapter is based.

We will not need to require IR explicitly :

Lemma 15.7. If $\phi : B \to \mathbb{R}^2$ is individually monotonic and Pareto optimal, then it is individually rational.

Proof. Let $\phi : B \to \mathbb{R}^2$ satisfy IM and PO. Let $S \in B$. Then $S_+ \subset S$ (cf. Def. 9.10) and $h(S_+) = h(S)$. By applying IM twice, we have $\phi(S) \geqq \phi(S_+)$, hence $\phi(S) \geqq 0$ since $\phi(S_+) \geqq 0$ by PO. ∎

It will be convenient to introduce the following monotonicity property, which is related to, but weaker than IM (cf. Roth (1979,p.101)).

Definition 15.8. A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ is called *restrictedly monotonic* if, for all $S,T \in B$ with $S \subset T$ and $h(S) = h(T)$, we have $\phi(T) \geqq \phi(S)$.

(Restricted monotonicity : RM)

Lemma 15.9. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying PO and IM or RM. Then $\phi(S) = \phi(S_+)$ for any $S \in B$.

Proof. Straightforward (cf. the proof of Lemma 15.7). ∎

Lemma 15.10. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying PO and STI. Then ϕ satisfies RM if and only if it satisfies IM.

Proof. If ϕ satisfies IM, then, obviously, ϕ satisfies also RM. Suppose now that ϕ satisfies RM, and let $S,T \in B$ with $S \subset T$ and $h_2(S) = h_2(T)$. (Cf. Fig. 15.2.) We shall prove

(15.1) $\phi_1(T) \geqq \phi_1(S)$.

In view of Lemma 15.9 we may assume $S = S_+$ and $T = T_+$.

Let $V := \{x \in T : x_1 \leqq h_1(S)\} \in B$. Then $h(V) = h(S)$ and $S \subset V$, hence, by RM, we obtain

(15.2) $\phi(V) \geqq \phi(S)$.

Let $\alpha := h_1(T)h_1(S)^{-1}$. By STI, $\phi((\alpha,1)V) = (\alpha\phi_1(V),\phi_2(V))$. Since $h(T) = h((\alpha,1)V)$ and $T \subset (\alpha,1)V$, we have by RM : $\phi((\alpha,1)V) \geqq \phi(T)$, hence

(15.3) $\phi_2(V) \geqq \phi_2(T)$.

Since $\phi(V) \in P(T)$ and $\phi(T) \in P(T)$, (15.3) implies

(15.4) $\phi_1(V) \leqq \phi_1(T)$.

Now we obtain (15.1) from combining (15.2) and (15.4). ∎

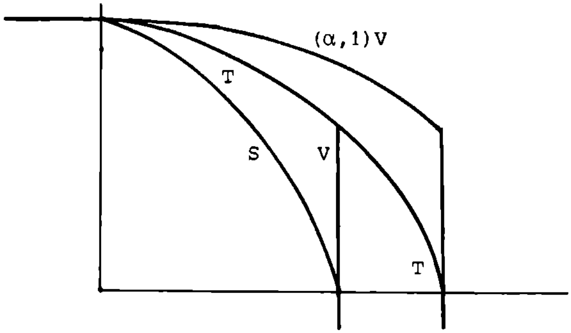


Figure 15.2.

We next define a family of 2-person bargaining solutions, and we shall prove that this family is the family of all solutions satisfying PO, STI, and IM.

Definition 15.11. A *monotonic curve* is a map $\lambda : [1,2] \to \mathrm{conv}\{(1,0),(0,1),(1,1)\}$ which has the following property :

(15.5) For all $s,t \in [1,2]$ with $s \leqq t$ we have $\lambda(s) \leqq \lambda(t)$ and $\lambda_1(s) + \lambda_2(s) = s$.

By Λ, we denote the family of all monotonic curves.

Note that $\lambda \in \Lambda$ is a continuous map : for $s,t \in [1,2]$, and $\|x\|_1 := |x_1| + |x_2|$ for $x \in \mathbb{R}^2$, we have $\|\lambda(s) - \lambda(t)\|_1 = |\lambda_1(s) - \lambda_1(t)| + |\lambda_2(s) - \lambda_2(t)| = |(\lambda_1(s) + \lambda_2(s)) - (\lambda_1(t) + \lambda_2(t))| = |s - t|$. Note further that, for every $S \in B$, the set $P(S) \cap \{\lambda(t) : t \in [1,2]\}$ contains exactly one point, which renders the following definition correct.

Definition 15.12. For every $\lambda \in \Lambda$, the 2-person bargaining solution $\pi^\lambda : B \to \mathbb{R}^2$ is defined by :

$\{\pi^\lambda(S)\} := P(S) \cap \{\lambda(t) : t \in [1,2]\}$ for every S with $h(S) = (1,1)$, and

$\pi^\lambda(S) := h(S)\pi^\lambda((h_1(S)^{-1},h_2(S)^{-1})S)$ otherwise.

We call π^λ *the solution corresponding to (the monotonic curve)* λ.

Proposition 15.13. For every $\lambda \in \Lambda$, π^λ satisfies PO, STI, and RM.

Proof. π^λ satisfies PO and STI by definition. Let $S,T \in B$ with $S \subset T$ and $h(S) = h(T) =: h$. Let $s,t \in [1,2]$ be such that $\pi^\lambda(S) = h\lambda(S)$ and $\pi^\lambda(T) = h\lambda(t)$. If $s > t$, then by (15.5) :

$h\lambda(s) \geqq h\lambda(t)$ and $h\lambda(s) \neq h\lambda(t)$, in contradiction with $S \subset T$ and PO of π^λ. So $s \leqq t$ and $\pi^\lambda(S) = h\lambda(s) \leqq h\lambda(t) \leqq \pi^\lambda(T)$. So π^λ satisfies RM. ∎

Note that the Kalai - Smorodinsky solution KS corresponds to the curve $\lambda \in \Lambda$ with $\lambda(t) = (\frac{1}{2}t, \frac{1}{2}t)$ for each $t \in [1,2]$; the dictator solutions D^1 and D^2 correspond to the curves λ' and λ'' in Λ with $\lambda'(t) = (1,t-1)$ and $\lambda''(t) = (t-1,1)$ for every $t \in [1,2]$.

We now prove the converse of Proposition 15.13.

Proposition 15.14. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution satisfying PO, STI and RM. Then $\phi = \pi^\lambda$ for some $\lambda \in \Lambda$.

Proof. (Cf. Fig. 15.3.) We first construct a curve $\lambda \in \Lambda$ with the aid of ϕ. To this end, let for every $t \in [1,2]$

$V(t) := \text{comv}\{(1,0),(0,1),(1,t-1),(t-1,1)\} \in B$. Define $\lambda : [1,2] \to \mathbb{R}^2$ by $\lambda(t) := \phi(V(t))$ for every $t \in [1,2]$. For $1 \leqq s \leqq t \leqq 2$ we have $\lambda(s) = \phi(V(s)) \leqq \phi(V(t)) = \lambda(t)$ by RM. Further, for each $t \in [1,2]$, $\lambda(t) \in P(V(t)) = \text{conv}(\{(1,t-1),(t-1,1)\})$, so $\lambda_1(t) + \lambda_2(t) = t$. Hence $\lambda \in \Lambda$, and

(15.6) $\phi(V(t)) = \pi^\lambda(V(t))$ for every $t \in [1,2]$.

We want to prove that $\phi = \pi^\lambda$. Let $S \in B$ with $h(S) = (1,1)$. In view of STI, it is sufficient to show $\phi(S) = \pi^\lambda(S)$ for such an S. Let now $s := \pi_1^\lambda(S) + \pi_2^\lambda(S)$. Let $W := \text{comv}(\{(0,1),(1,0),\pi^\lambda(S)\})$. Then

(15.7) $\pi^\lambda(S) = \pi^\lambda(W) = \pi^\lambda(V(s)) = \phi(V(s)) \in P(S) \cap P(W) \cap P(V(s))$, where the last equality follows from (15.6). Since $W \subset V(s)$, and $h(W) = h(V(s)) = (1,1)$, we obtain by RM, PO and (15.7) :

(15.8) $\phi(V(s)) = \phi(W)$.

Since $W \subset S$ and $h(W) = h(S)$, we have by (15.8), (15.7), and RM :

(15.9) $\phi(W) = \phi(S)$.

Combining (15.7) - (15.9), we conclude that $\phi(S) = \pi^\lambda(S)$. ∎

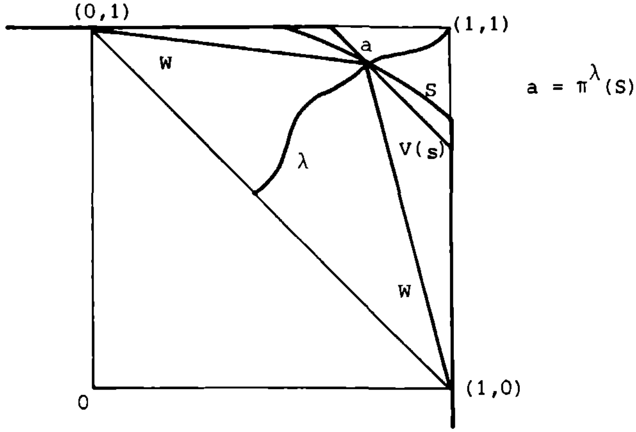


Figure 15.3.

We can now readily prove :

Theorem 15.15. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution. Then ϕ satisfies PO, STI, and IM, if and only if $\phi = \pi^\lambda$ for some $\lambda \in \Lambda$.

Proof. Follows from Lemma 15.10 and Propositions 15.13 and 15.14. ∎

Note that Theorem 15.4 follows from Theorem 15.15. In the following corollary, we link Theorem 15.15 and Theorem 11.8.

Corollary 15.16. There are exactly two solutions on B satisfying PO, STI, IM, and IIA, namely the dictator solutions D^1 and D^2.

Proof. Theorem 29.17 for the case n=2. ∎

We might view Corollary 15.16 as a kind of impossibility result, namely if we regard PO, STI, IM and IIA as desirable properties for a 2-person bargaining solution to have, and if we also wish to rule out dictatorial solutions.

In section 21 we shall prove a more general result (a characterization of a family of individually monotonic "multisolutions") which implies Theorem 15.15. Thomson (1984) accounts for the nonsymmetry of solutions in a family of IM-solutions which is related to the family in Theorem 15.15, by considering replications of 2-person bargaining games. Finally, we remark that the Kalai -

Smorodinsky solution also shows up in experiments, see Crott (1971).

16. GLOBAL INDIVIDUAL MONOTONICITY

In the previous section, we have characterized a family of individually monotonic solutions : these solutions depend for a given bargaining game on two important points - reference points -, namely the disagreement outcome and the utopia point. With some modifications, similar techniques can be used to characterize "monotonic" solutions, w.r.t. other reference points. In this section, we will show this for a family of solutions depending on the global utopia point : see Def. 9.10. Thereby we extend a result of Kalai and Rosenthal (1978).

Definition 16.1. A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ is called *globally individually monotonic* if, for all $i,j \in \{1,2\}$ with $i \neq j$ and $S,T \in B$ with $S \subset T$ and $g_i(S) = g_i(T)$, we have $\phi_j(S) \leq \phi_j(T)$.
(Global individual monotonicity : GIM)

Example 16.2. Let the solution $\phi : B \to \mathbb{R}^2$ be equal to the solution ϕ in Example 15.5. Let the solution $\psi : B \to \mathbb{R}^2$ be defined by : $\psi(S) \in P(S)$ has maximal second coordinate, for every $S \in B$. Then ϕ satisfies IR, WPO, STI, and GIM, but not PO; ψ satisfies PO, STI, GIM, but not IR.

We wish to restrict attention to (STI,GIM) solutions which are (strongly) Pareto optimal and individually rational; Example 16.2 shows that IR is not implied by PO, nor PO by IR, in the presence of STI and GIM. (We note, however, that Remark 15.6 still applies if we replace IM by GIM.) The following two lemmas correspond to Lemmas 15.9 and 15.10 in the previous section.

Lemma 16.3. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution satisfying PO, IR, and GIM. Then $\phi(S) = \phi(T)$ for any S and T in B with $g(S) = g(T)$ and $S_+ = T_+$.

Proof. Choose $\alpha \in \mathbb{R}$ so small that $(\alpha, g_2(S)) = (\alpha, g_2(T)) \in S \cap T$, and choose $\beta \in \mathbb{R}$ so small that $(g_1(S), \beta) = (g_1(T), \beta) \in S \cap T$. Let

$V := \mathrm{comv}(\{(\alpha,g_2(S)),(g_1(S),\beta)\} \cup S_+)$, then $V \in B$, $V \subset S$, $V \subset T$, $g(V) = g(S) = g(T)$, and $V_+ = S_+ = T_+$. By applying IR, PO, and GIM twice, we find that $\phi(V) = \phi(S)$. Similarly, $\phi(V) = \phi(T)$. So $\phi(S) = \phi(T)$. ■

Lemma 16.4. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution satisfying PO, IR, and STI. Then ϕ satisfies GIM if and only if $\phi(S) \leqq \phi(T)$ for all S and T in B with $S \subset T$ and $g(S) = g(T)$.

Proof. Analogous to the proof of Lemma 15.10; we note that, whereas there we needed Lemma 15.9, here we do not need Lemma 16.3. ■

We want to describe all 2-person bargaining solutions $\phi : B \to \mathbb{R}^2$ which have the properties IR, PO, STI, and GIM. Therefore we introduce a family of curves resembling the family Λ in the previous section.

Definition 16.5. By Θ, we denote the family of maps $\theta : [0,2] \to \mathrm{conv}(\{(0,0),(1,0),(0,1),(1,1)\})$ which satisfy

(16.1) For all $s,t \in [0,2]$: $\theta(s) \leqq \theta(t)$ if $s \leqq t$, and $\theta_1(s) + \theta_2(s) = s$.

Similarly as in section 15 for $\lambda \in \Lambda$, it follows from (16.1) that $\theta \in \Theta$ is a continuous map. And note that, for every $S \in B$, the set $P(S) \cap \{\theta(t) : t \in [0,2]\}$ contains exactly one element.

Definition 16.6. For $\theta \in \Theta$, the solution $\psi^\theta : B \to \mathbb{R}^2$ is defined by : for every $S \in B$ with $g(S) = (1,1)$, $\{\psi^\theta(S)\} := P(S) \cap \{\theta(t) : t \in [0,2]\}$, and $\psi^\theta(S) := g(S)\psi^\theta((g_1(S)^{-1},g_2(S)^{-1})S)$ otherwise. We call ψ^θ *the solution corresponding to* θ.

Proposition 16.7. For every $\theta \in \Theta$, ψ^θ satisfies IR, PO, STI, and GIM.

Proof. ψ^θ satisfies IR, PO, and STI, by definition. GIM follows easily with the aid of Lemma 16.4. ■

Proposition 16.8. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution satisfying IR, PO, STI, and GIM. Then $\phi = \psi^\theta$ for some $\theta \in \Theta$.

Proof. The proof is very much like the proof of Proposition 15.14. First, we construct a map θ with the aid of ϕ. To this end, let, for each $t \in (0,1)$, $L(t) := \mathrm{comv}(\{(-1,1),(0,t),(t,0),(1,-1)\})$, and let, for each $t \in [1,2]$,

$L(t) := \text{comv}(\{(t-1,1),(1,t-1)\})$. Define $\theta : [0,2] \to \mathbb{R}^2$ by $\theta(0) := (0,0)$ and $\theta(t) := \phi(L(t))$ if $t \in (0,2]$. Similarly as in the proof of Proposition 15.14, now using Lemma 16.4, we have $\theta \in \Theta$ and

(16.2) $\phi(L(t)) = \psi^{\theta}(L(t))$ for each $t \in (0,2]$.

Let $S \in B$. We shall prove

(16.3) $\phi(S) = \psi^{\theta}(S)$.

In view of STI, we may suppose $g(S) = (1,1)$. Let $s := \psi_1^{\theta}(S) + \psi_2^{\theta}(S)$.
By definition of s and of ψ^{θ}, and by PO, we have

(16.4) $\psi^{\theta}(S) = \psi^{\theta}(L(s)) \in P(S) \cap P(L(s)) \subset P(S \cap L(s))$.

In view of Lemma 16.4, $\phi(L(s)) \geqq \phi(S \cap L(s))$. In view of PO, (16.2) and (16.4), we obtain $\phi(L(s)) = \phi(S \cap L(s))$. By Lemma 16.4 again, $\phi(S) \geqq \phi(S \cap L(s)) = \phi(L(s))$, hence

(16.5) $\phi(S) = \phi(L(s))$,

since $\phi(L(s)) \in P(S)$ by (16.2) and (16.4).
Now (16.3) follows if we combine (16.2), (16.4), and (16.5). ■

Propositions 16.7 and 16.8 now lead to the main result of this section.

Theorem 16.9. A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ satisfies IR, PO, STI, and GIM, if and only if $\phi = \psi^{\theta}$ for some $\theta \in \Theta$.

We call the solution $KR : B \to \mathbb{R}^2$ which corresponds to $\theta \in \Theta$ with $\theta(t) := (\frac{1}{2}t, \frac{1}{2}t)$ for each $t \in [0,2]$, the *Kalai - Rosenthal solution*, since this solution was proposed as an arbitration scheme in Kalai and Rosenthal (1978). It is the unique symmetric member of $\{\psi^{\theta} : \theta \in \Theta\}$.
Note further that the dictator solutions D^1 and D^2 correspond to θ' and θ'' in Θ, respectively, where $\theta'(t) = (t,0)$ and $\theta''(t) = (0,t)$ for each $t \in [0,1]$, and $\theta'(t) = (1,t-1)$ and $\theta''(t) = (t-1,1)$ for each $t \in [1,2]$.

We conclude with the following corollary of the Theorems 11.8 and 16.9, and Corollary 15.16.

Corollary 16.10. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution satisfying IR, PO, and STI. Then $\phi \in \{D^1, D^2\}$ if and only if ϕ has at least two of the three properties IIA, IM, and GIM.

Proof. If $\phi = D^1$ or $\phi = D^2$, then ϕ satisfies IIA, IM, and GIM (e.g., Corollary 15.16 and Theorem 16.9). Let now ϕ satisfy IR, PO, and STI. If ϕ satisfies

IIA as well as IM, then $\phi = D^1$ or $\phi = D^2$ by Corollary 15.16.

Suppose ϕ satisfies IIA and GIM. Let $0 \leqq t \leqq 1$ with $(t,1-t) = \phi(\Delta)$. (Cf. Remark 12.10.) By STI, $\phi(\lambda\Delta) = (\lambda t, \lambda(1-t))$ for every $\lambda \in \mathbb{R}$, $\lambda > 0$. So by PO, IR, and IIA applied to $\lambda\Delta \subset \operatorname{comv}\{(\lambda-1,1),(1,\lambda-1)\}$, we have $\phi(\operatorname{comv}\{(\lambda-1,1),(1,\lambda-1)\}) = (\lambda t, \lambda(1-t))$ for every $\lambda \in \mathbb{R}$, $\lambda > 0$. This means, in view of Theorem 16.9, that ϕ corresponds to a curve $\theta \in \mathcal{O}$ with

(16.6) $\theta(s) = (ts,(1-t)s)$ for $0 \leqq s \leqq 1$.

Consider $V := \operatorname{comv}(\{(-1,1),(1,0)\})$. By IR, PO, STI, and IIA applied to $(1,\frac{1}{2})\Delta \subset V$, we obtain $\phi_1(V) = t$. By (16.6), we have $\phi_1(V) = \frac{t}{2-t}$. So either $t=1$ or $t=0$, which, in view of Theorem 11.8, implies $\phi = D^1$ or $\phi = D^2$, respectively.

Finally, suppose ϕ satisfies IM and GIM. Let again $\phi(\Delta) = (t,1-t)$ for some $0 \leqq t \leqq 1$. In the same way as in the previous part of the proof (replace IIA by IM everywhere) we have that either $t=1$ or $t=0$. But then it follows with the aid of Theorem 16.9, that $\phi = D^1$ or $\phi = D^2$. ■

ADDITIVITY PROPERTIES

The so-called Super-Additivity property of 2-person bargaining solutions was proposed by Perles and Maschler (1981). They used the property to characterize a, by that time new, solution. Under a different name, it also occurs in Myerson (1981).

In this chapter, we shall consider the so-called (weak) super-additivity and restricted additivity properties. As main results, we obtain new characterizations of the (extended) family of proportional solutions of Kalai (Kalai (1977)), and of the family of Nash solutions N; further, we give an interpretation of the use of additivity properties, which is different from the interpretation of the super-additivity property provided by the authors mentioned in the first paragraph. Our interpretation rests on the possibility that, sometimes, simultaneous bargaining over more than one issue has advantages for both bargainers over dealing with these issues separately. In this setting, we shall see that the result of section 6 provides a utility-theoretic foundation for the use of additivity properties.

The plan of this chapter is as follows. In section 17, a relation between simultaneity of issues and additivity in bargaining is established. Section 18 characterizes the above mentioned family of proportional solutions with the aid of the so-called partial super-additivity property. Section 20 provides a new characterization of the family of Nash solutions N with the aid of the so-called restricted additivity property, and is preceded by a short section (19) on continuity properties of solutions. This chapter closely follows Peters (1985 ,1986).

17. SIMULTANEITY OF ISSUES AND ADDITIVITY IN BARGAINING

Suppose, two parties are facing several (separate) bargaining situations, on (possibly quite) different issues. Handling these situations one by one may yield both parties only small profits. Bargaining, however, over these issues simultaneously, may yield both parties larger total profits, thus reflecting more properly their perhaps strong interests in some of these issues. The following simple example illustrates this.

Mr. X and his wife each have a ticket for a magnificent movie, but, unfortunately, these tickets are not valid for the same show (there are two shows which take place on different evenings). Now, for each of the two shows for which one of the tickets is valid, there are three alternatives : (a) the ticket-holder watches the movie leaving his/her partner at home, which gives him/her 6 units of utility and his/her partner -2 units; (b) they both stay at home, but with the ticket-holder grudging the whole evening : 0 utility for both (the disagreement alternative); (c) they both stay at home and play some card-game : 0 utility for the ticket-holder and 1 unit of utility for the partner. If we suppose for a moment that these utilities are additive, then Mr. X as well as his wife do very well by each one using his/her ticket and receiving a net utility of 4.

Modelling the above example with the aid of bargaining games, we may call Mr. X player 1 and Mrs. X player 2. There are two bargaining games to be played, corresponding to the two evenings on which the shows take place. If we suppose that Mr. X's ticket is valid for the first show, then on this first evening Mr. X and his wife play a bargaining game $S = \text{comv}\{(6,-2),(0,0),(0,1)\} \in B$ where these three points correspond to the alternatives (a), (b), and (c) in the previous paragraph and where we assume that lotteries between these alternatives are possible. Similarly, on the second evening they play $T = \text{comv}\{(-2,6),(0,0),(1,0)\} \in B$. Suppose further that Mr. X and his wife agree to let some individually rational, Pareto optimal and symmetric bargaining solution $\phi : B \to \mathbb{R}^2$ solve their conflicts. Then note that $\phi_i(S) + \phi_i(T) \leqq 3$ for $i \in \{1,2\}$, whereas $\phi(\{s + t : s \in S, t \in T\}) = (4,4)$. So it is clearly advantageous for both players to play both games simultaneously, provided we can find a way to let $\{s + t : s \in S, t \in T\} \in B$ represent the simultaneous bargaining game; to this end, we use the additive utility model of section 6, as follows.

Let $\Sigma \subset BS$ be a family of 2-person bargaining situations, such that, for any $\Gamma = \langle A,\bar{a},u^1,u^2\rangle$ and $\Gamma' = \langle B,\bar{b},v^1,v^2\rangle$ in Σ, both players have preference relations (representable by vNM utility functions) also on $L(A \times B)$, and such that conditions (6.2) and (6.4) hold, with $\bar{a}$ and $\bar{b}$ for $a°$ and $b°$, respectively, in (6.4), and such that Remark 6.2 applies. Then Theorem 6.1 and Remark 6.2 tell us that the players' preference relations on $L(A \times B)$ are representable by vNM utility functions w^i with $w^i(a,b) = u^i(a) + v^i(b)$ for each $i \in \{1,2\}$, all $a \in A$, and all $b \in B$.

For Γ, Γ' as above, we call $\Gamma \times \Gamma' := \langle A \times B, (\bar{a},\bar{b}), w^1, w^2 \rangle$ the *corresponding simultaneous bargaining situation.* Note that $\Gamma \times \Gamma' \in BS$. The following condition expresses that simultaneous bargaining may be more profitable for both bargainers than separate bargaining, given a bargaining solution $\phi : B \to \mathbb{R}^2$.

(17.1) If $\ell \in alt(\phi,\Gamma)$, $m \in alt(\phi,\Gamma')$, and $n \in alt(\phi,\Gamma \times \Gamma')$, then, for $i = 1,2$, $w^i(n) \geqq u^i(\ell) + v^i(m)$.

Equivalently :

(17.2) $\phi(S_{\Gamma \times \Gamma'}) \geqq \phi(S_\Gamma) + \phi(S_{\Gamma'})$.

In view of the additivity of the utility functions, we may write, instead of (17.2) :

(17.3) $\phi(S_\Gamma + S_{\Gamma'}) \geqq \phi(S_\Gamma) + \phi(S_{\Gamma'})$.

(In general, for $S,T \subset \mathbb{R}^k$, we denote $S + T := \{s + t \in \mathbb{R}^k : s \in S, t \in T\}$.)

More generally, we formulate the following property.

Definition 17.1. We call a bargaining solution $\phi : B \to \mathbb{R}^2$ *super-additive* if $\phi(S + T) \geqq \phi(S) + \phi(T)$ for all $S, T \in B$.
(Super-additivity : SA)

Remark 17.2. Actually, we can find Σ as above such that for every $S \in B$ there is $\Gamma \in \Sigma$ with $S = S_\Gamma$: namely, let Σ consist of all trivial bargaining situations corresponding to bargaining games in B (cf. Example 9.6), with the sets of riskless alternatives extended with $(-1,-1)$, for instance, to guarantee that Remark 6.2 holds.

In the remainder of this chapter, we shall concern ourselves with studying solutions satisfying the super-additivity property or variations on it. Perles and Maschler (1981) have shown that there exists a unique bargaining solution having, besides some other properties, the super-additivity property. There are two main differences between their approach and ours. First, they provide an interpretation of the use of the SA property which is different from ours, and which is given by the following observation.

Observation 17.3. Let $\phi : B \to \mathbb{R}^2$ be a two-person bargaining solution satisfying SA and HOM (see Def. 10.4). For any game consisting of a lottery on two games R and S in B, players who obey ϕ will prefer to reach an agreement before the outcome of the lottery is available.

Proof. Let (p,1-p) be the distribution of the lottery, w.l.o.g. $0 < p < 1$. If the players reach an agreement immediately, it must be $\phi(T)$, where $T = pR + (1-p)S$. By HOM and SA,

(17.4) $\phi(T) \geqq p\phi(R) + (1-p)\phi(S)$.

The right hand side of (17.4) is the expectation of the players from a delayed agreement. ■

Secondly, Perles and Maschler restrict their attention to games in the class B_+ (cf. Definition 9.10) where no player can commit himself to a feasible outcome which is not individually rational for the other player (i.e. an outcome with at least one coordinate negative). Indeed, if one feels that one is actually dealing with noncooperative Nash bargaining games (Perles and Maschler (1981), p.167), then this restriction to B_+ is justified. Recall the example of Mr. X and his wife at the beginning of this section. The outcome, there, of the sumgame, (4,4), can only be achieved by the sum (6,-2) + (-2,6). This means that in one game player 1 can commit himself to (6,-2), whereas in the other game player 2 can commit himself to (-2,6). In a noncooperative setting, such commitments would be impossible : we are stuck in a prisoner's dilemma (cf. Example 1.1). Yet in a cooperative setting, where binding agreements are possible, these commitments lead to a net utility profit of 4 for both players. We shall assume such a cooperative setting. Perles and Maschler have already indicated that their solution cannot be extended to B. This will also follow as a corollary of the results in the next section.

18. (PARTIAL) SUPER-ADDITIVITY

In this section, we single out a family of super-additive solutions (cf. Definition 17.1) with the aid of the properties weak Pareto optimality, homogeneity, and the following property.

Definition 18.1. We call the 2-person bargaining solution *partially super-additive* if $\phi(S + T) \geq \phi(S)$ and $\phi(S + T) \geq \phi(T)$ for all $S, T \in B$. (Partial super-additivity : PSA)

We prefer the adjective "partial" to "weak" since PSA is not implied by SA alone. However,

Lemma 18.2. If $\phi : B \to \mathbb{R}^2$ satisfies IR and SA, then it also satisfies PSA.

Proof. Straightforward. ∎

We next define a family of bargaining solutions which, for strictly positive weight vectors, were introduced in Kalai (1977).

Definition 18.3. For every $p \in \mathbb{R}^2_+$ with $p_1 + p_2 = 1$, the bargaining solution $E^p : B \to \mathbb{R}^2$ is defined by

$$\{E^p(S)\} = W(S) \cap \{\alpha p : \alpha \in \mathbb{R}, \alpha > 0\} \text{ for every } S \in B.$$

E^p is called the *egalitarian* or *proportional solution with weight vector* p.

Our main result in this section is the following theorem.

Theorem 18.4. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution. Then ϕ satisfies WPO, HOM, and PSA, if and only if it is proportional.

The proof of this theorem will make use of the following three lemmas.

Lemma 18.5. Let $\phi : B \to \mathbb{R}^2$ be a solution satisfying HOM and PSA, let $S \in B$, and let $r \in \mathbb{R}^2_{++}$. Then

(i) if $r \in \text{int}(S)$, then $\phi(S) \geq \phi(\text{com}\{r\})$,

(ii) $\phi(\text{com}\{r\}) \geq 0$.

Proof. (i) $S = \text{com}\{r\} + \{x - r : x \in S\}$, so, if $r \in \text{int}(S)$, we have by PSA : $\phi(S) \geq \phi(\text{com}\{r\})$.

(ii) By HOM : $\phi(\text{com}\{\frac{1}{2}r\}) = \frac{1}{2}\phi(\text{com}\{r\})$, and, since $\frac{1}{2}r \in \text{int}(\text{com}\{r\})$, by (i) : $\phi(\text{com}\{r\}) \geq \phi(\text{com}\{\frac{1}{2}r\})$. So $\phi(\text{com}\{\frac{1}{2}r\}) \geq \frac{1}{2}\phi(\text{com}\{\frac{1}{2}r\})$. Hence $\phi(\text{com}\{r\}) = 2\phi(\text{com}\{\frac{1}{2}r\}) \geq 0$. ∎

Corollary 18.6. Every homogeneous and partially super-additive bargaining solution $: B \to \mathbb{R}^2$ is individually rational.

Proof. Follows immediately from Lemma 18.5. ■

Let now L be the set $\{p \in \mathbb{R}^2_{++} : p_1 + p_2 = 1\}$.

Lemma 18.7. Let $\phi : B \to \mathbb{R}^2$ be a solution satisfying HOM, PSA, and WPO. Then : Either (i) $\phi(\text{com}\{p\}) = p$ for some $p \in L$, or (ii) $\phi_2(\text{com}\{p\}) < p_2$ for all $p \in L$, or (iii) $\phi_1(\text{com}\{p\}) < p_1$ for all $p \in L$.

Proof. Suppose (i) does not hold. Let $L^1 := \{p \in L : \phi_1(\text{com}\{p\}) < p_1\}$, $L^2 := \{p \in L : \phi_2(\text{com}\{p\}) < p_2\}$, and suppose that $L^1 \neq \emptyset$, $L^2 = \emptyset$.
Let $p^1 \in L^1$, $p^2 \in L^2$. We show

(18.1) $\quad p_1^1 \geq p_1^2$.

Suppose (18.1) does not hold, i.e. $p_1^1 < p_1^2$ and $p_2^1 > p_2^2$. Let then $q \in \mathbb{R}^2_{++}$ be defined by $q_1 := \frac{1}{2}(p_1^1 + \phi_1(\text{com}\{p^1\}))$, $q_2 := \frac{1}{2}(p_2^2 + \phi_2(\text{com}\{p^2\}))$.
Then $q \in \text{int}(\text{com}\{p^1\})$, so by Lemma 18.5 (i), $\phi_1(\text{com}\{p^1\}) \geq \phi_1(\text{com}\{q\})$.
Similarly, $\phi_2(\text{com}\{p^2\}) \geq \phi_2(\text{com}\{q\})$. Altogether we obtain $q > \phi(\text{com}\{q\})$, in contradiction with WPO. So (18.1) must hold.
From our assumptions : (i) does not hold, $L^1 \neq \emptyset$, $L^2 \neq \emptyset$, we conclude, with the aid of (18.1), that there exists a $\bar{p} \in L$ such that for all $p \in L$ with $p_1 < \bar{p}_1$ we have $p \in L^2$, and for all $p \in L$ with $p_1 > \bar{p}_1$ we have $p \in L^1$.
The proof of the lemma is finished, by contradiction, if we show

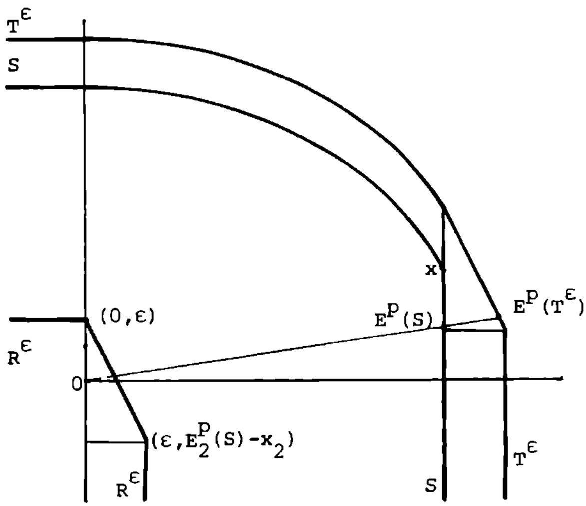
(18.2) $\quad \phi(\text{com}\{\bar{p}\}) = \bar{p}$.

For any $0 < x < \bar{p}$ with $x_1 x_2^{-1} < \bar{p}_1 \bar{p}_2^{-1}$, we have, by HOM, WPO, and the fact that $\alpha x \in L^2$ for some $\alpha > 0$, that $\phi_1(\text{com}\{x\}) = x_1$. Similarly, $\phi_2(\text{com}\{y\}) = y_2$ for any $0 < y < \bar{p}$ with $y_1 y_2^{-1} > \bar{p}_1 \bar{p}_2^{-1}$. So by Lemma 18.5 (i) we conclude that (18.2) holds. ■

Lemma 18.8. Let $\phi : B \to \mathbb{R}^2$ be a solution satisfying HOM, PSA, and WPO. Let $p \in L$ such that $\phi(\text{com}\{p\}) = p$ if (i) in Lemma 18.7 holds, let $p = (0,1)$ if (iii) there holds, and let $p = (1,0)$ if (ii) there holds.
Then $\phi = E^p$.

Proof. Let $S \in B$. First suppose $E^p(S) \in P(S)$. If $p > 0$,

$\phi(S) \geq \phi(\mathrm{com}\{(1-\varepsilon)E^p(S)\}) = (1-\varepsilon)E^p(S)$ for $1 > \varepsilon > 0$ by Lemma 18.5 (i) and HOM, so the proof is finished by letting ε approach 0. If $p = (1,0)$, then take a sequence $r^1, r^2, \ldots$ in $\mathrm{int}(S) \cap \mathbb{R}^2_{++}$ converging to $E^p(S)$. Then again $\phi(S) \geq \phi(\mathrm{com}\,\{r^i\})$ for each $i = 1,2,\ldots$, so $\phi_1(S) \geq \phi_1(\mathrm{com}\{r^i\}) = r^i_1$ for each $i = 1,2,\ldots$, hence $\phi_1(S) \geq E^p_1(S)$. We conclude that $\phi(S) = E^p(S)$. By a similar argument, $\phi(S) = E^p(S)$ if $p = (0,1)$.
Suppose now that $E^p(S) \notin P(S)$. We assume (the other case is similar) that there exists $x \in P(S)$ with $x_1 = E^p_1(S)$. Given $\varepsilon > 0$, let $R^\varepsilon \in B$ be defined by $R^\varepsilon := \mathrm{comv}\{(\varepsilon, E^p_2(S) - x_2), (0,\varepsilon)\}$. Let $T^\varepsilon = S + R^\varepsilon$. (See Fig. 18.1.) Note that $E^p(T^\varepsilon) \in P(T^\varepsilon)$. By the first part of this proof, $\phi(T^\varepsilon) = E^p(T^\varepsilon)$. If ε approaches 0, $E^p(T^\varepsilon) = \phi(T^\varepsilon)$ converges to $E^p(S)$, and by PSA, $\phi(T^\varepsilon) \geq \phi(S)$ for all ε, so $E^p(S) \geq \phi(S)$. If $p = (1,0)$ the proof is complete. If $p > 0$, then also the proof is complete if we note that $\phi(S) \geq E^p(S)$ by the argument in the third sentence of the proof. ■



<u>Figure 18.1.</u>

An immediate consequence of Lemma 18.8 is that, if, for ϕ there, $\phi(\mathrm{com}\{p\}) = p$ for some $p \in L \cup \{(1,0),(0,1)\}$, then this p is unique. The proof of Theorem 18.4 is now straightforward.

Proof of Theorem 18.4. If ϕ satisfies the three properties in the theorem, then ϕ is proportional in view of Lemma 18.8. And it is straightforward to verify that a proportional solution has these properties. ∎

Note that PSA is implied by SA and IR combined (Lemma 18.2), and that every proportional solution satisfies SA. So the following corollary is immediate.

Corollary 18.9. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution. Then ϕ has the properties WPO, HOM, IR, and SA, if and only if ϕ is proportional.

Perles and Maschler (1981) require their solution to have (on B_+) the properties PO, STI, SA, SYM, and some continuity property (on continuity properties : see the next section). We must add IR if we consider solutions defined on B. Perles and Maschler call their solution the "Super-Additive solution".

Corollary 18.10. There does not exist a solution $\phi : B \to \mathbb{R}^2$ with the properties STI, SA, PO, and IR. In particular, there does not exist a Super-Additive solution on B.

Proof. Straightforward from Corollary 18.9. ∎

If we assume the players to have vNM utility functions, then we may consider it a drawback for a solution not to have the scale transformation invariance property (cf. (4.2)). There are only two scale transformations invariant proportional solutions : this observation leads to the following corollary immediately.

Corollary 18.11. The only two solutions on B satisfying WPO, STI, IR, and SA, are the proportional solutions $E^{(1,0)}$ and $E^{(0,1)}$.

By excluding "tyrannical" solutions (i.e. $E^{(1,0)}$, $E^{(0,1)}$), Corollary 18.11 may be formulated as an "impossiblity result". In section 20, we will weaken the super-additivity property, and obtain an alternative characterization of the family of nonsymmetric Nash solutions N.

Myerson (1981) and Kalai (1977) also give characterizations of the family of proportional solutions (the two "tyrannical" solutions excluded) using different lists of properties. Kalai and Samet (1985) give an axiomatic characterization of "proportional solutions" for the general case of n-person games without sidepayments.

19. CONTINUITY PROPERTIES

Let K denote the family of those compact convex subsets of $\mathbb{R}^2$ which contain the origin and a point with both coordinates strictly positive. We provide K with the Hausdorff *metric* $d_H^K : K \times K \to \mathbb{R}$ defined by

$$d_H^K(S,T) := \inf\{\varepsilon > 0 : S \subset B_\varepsilon(T) \text{ and } T \subset B_\varepsilon(S)\}$$

for all $S, T \in K$, where $B_\varepsilon(S) := \{x \in \mathbb{R}^2 : \min_{s \in S} \|x - s\|_\infty \leq \varepsilon\}$.

Further, let $\psi : K \to \mathbb{R}^2$ be a map such that $\psi(S) \in S$ for every $S \in K$. We call ψ *continuous* if, for every sequence $S, S_1, S_2, \ldots$ in K with $\lim_{n\to\infty} d_H^K(S_n, S) = 0$, we have $\lim_{n\to\infty} \psi(S_n) = \psi(S)$.

We now also provide the class of bargaining games B with a metric denoted d_H^B and defined analogous to d_H^K.

(The reader may verify that $d_H^B : B \times B \to \mathbb{R}$ is a metric indeed.) *Continuity of a bargaining solution* $\phi : B \to \mathbb{R}^2$ is defined similarly as continuity of $\psi : K \to \mathbb{R}^2$ above. Jansen and Tijs (1983) show that many well-known maps ("bargaining solutions") ψ defined on K ("bargaining games") as above, are continuous. The following lemma relates continuity of such a ψ on K to continuity of a bargaining solution ϕ on B.

Lemma 19.1. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution and let $\psi : K \to \mathbb{R}^2$ be a map with $\psi(S) = \psi(S \cap \mathbb{R}^2_+) \in S$ for every $S \in K$. Let further $\phi(\text{com}(S)) = \psi(S)$ for every $S \in K$. Then : ϕ is continuous if and only if ψ is continuous.

Proof. First, suppose ϕ is continuous. Let $S, S_1, S_2, \ldots \in K$ with $\lim_{n\to\infty} d_H^K(S_n, S) = 0$. Then also $\lim_{n\to\infty} d_H^B(\text{com}(S_n), \text{com}(S)) = 0$, hence $\lim_{n\to\infty} \phi(\text{com}(S_n)) = \phi(\text{com}(S))$, so $\lim_{n\to\infty} \psi(S_n) = \psi(S)$: ψ is continuous.

Next, suppose ψ is continuous. Let $S,S_1,S_2,\ldots \in B$ with $\lim_{n\to\infty} d_H^B(S_n,S) = 0$. For $T \in B$, let $m(T) \in \mathbb{R}^2$ be defined by

$m_1(T) := \min\{0,x_1 : (x_1,x_2) \in P(T)\}$ and

$m_2(T) := \min\{0,x_2 : (x_1,x_2) \in P(T)\}$. Now $\lim_{n\to\infty} d_H^K(S_n \cap \mathbb{R}_+^2, S \cap \mathbb{R}_+^2) = 0$,

hence $\lim_{n\to\infty} \psi(S_n \cap \mathbb{R}_+^2) = \psi(S \cap \mathbb{R}_+^2)$, hence

$\lim_{n\to\infty} \psi(\{x \in S_n : x \geq m(S_n)\}) = \psi(\{x \in S : x \geq m(S)\})$,

hence $\lim_{n\to\infty} \phi(S_n) = \phi(S)$: ϕ is continuous. ■

Remark 19.2. The condition "$\psi(S) = \psi(S \cap \mathbb{R}_+^2)$" in Lemma 19.1 is not a necessary condition. For instance, the map $\psi : K \to \mathbb{R}^2$ with definition similar to the definition of the Kalai - Rosenthal solution (see after Theorem 16.9) is continuous (Proposition 3.1 (i)(b) in Jansen and Tijs (1983)), whereas the Kalai - Rosenthal solution $KR : B \to \mathbb{R}^2$ is also continuous (proof left to the reader). Whether we can dispense with some condition of this kind altogether, is an open question.

By combining Lemma 19.1 with Propositions 3.1 (i)(a) and 3.2 in Jansen and Tijs (1983), we obtain :

Corollary 19.3. For every $t \in (0,1)$, the nonsymmetric Nash-solution $N^t : B \to \mathbb{R}^2$ is continuous. The dictator solutions $D^1,D^2 : B \to \mathbb{R}^2$ are discontinuous.

In the remainder of this section, we shall introduce a different metric on B such that, w.r.t. this metric, also the dictator solutions are continuous. This result will be used in the next section.

Definition 19.4. We call a bargaining solution $\phi : B \to \mathbb{R}^2$ *Pareto continuous* if, for every sequence $S,S_1,S_2,\ldots$ in B satisfying

(19.1) $\lim_{n\to\infty} d_H^K(\mathrm{conv}(P(S_n)),\mathrm{conv}(P(S))) = 0$,

we have $\lim_{n\to\infty} \phi(S_n) = \phi(S)$.

(Pareto continuity : PCO)

The following lemma is straightforward :

Lemma 19.5. If the bargaining solution $\phi : B \to \mathbb{R}^2$ is continuous, then it is Pareto continuous.

We conclude with :

Theorem 19.6. (a) For every $t \in (0,1)$, the nonsymmetric Nash solution $N^t : B \to \mathbb{R}^2$ is Pareto continuous. (b) The dictator solutions $D^1, D^2 : B \to \mathbb{R}^2$ are Pareto continuous.

Proof. (a) Follows from Corollary 19.3 and Lemma 19.5.

(b) Let $S, S_1, S_2, \ldots$ be a sequence in B satisfying (19.1). Then, in particular, $\lim_{n\to\infty} q(S_n) = q(S)$ where, for every $T \in B$, $q(T)$ is the point of $P(T)$ with maximal first coordinate. If $q(S) > 0$, then $\lim_{n\to\infty} q(S_n) = q(S)$ implies $\lim_{n\to\infty} D^1(S_n) = q(S) = D^1(S)$. If $q_2(S) < 0$, then there is an $M \in \mathbb{N}$ such that $D_2^1(S_n) = 0 (= D_2^1(S))$ for all $n \geq M$: so again $\lim_{n\to\infty} D^1(S_n) = D^1(S)$. The same argument holds if $q_2(S) = 0$ and $q_2(S_n) \leqq 0$ for all n larger than some $M' \in \mathbb{N}$; if $q_2(S) = 0$ and there is a subsequence $S_{n_1}, S_{n_2}, \ldots$ with $q(S_{n_i}) > 0$ for all $i \in \mathbb{N}$, then we have $\lim_{n\to\infty} D^1(S_n) = \lim_{i\to\infty} D^1(S_{n_i}) = \lim_{i\to\infty} q(S_{n_i}) = q(S) = D^1(S)$. We conclude that D^1 has the PCO property. Similarly one proves PCO for D^2. ■

20. RESTRICTED ADDITIVITY

In this section, we shall give an alternative characterization of the family N of nonsymmetric Nash solutions, with the aid of a weaker version of the super-additivity property. (For the following definition, recall from section 12 that we call an $S \in B$ smooth at $x \in S$ if S has a unique supporting line at x.)

Definition 20.1. A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ is called *restrictedly additive* if, for all S and T in B such that S and T are smooth at $\phi(S)$ and $\phi(T)$ respectively, and $\phi(S) + \phi(T) \in P(S + T)$, we have

$\phi(S + T) = \phi(S) + \phi(T)$.
(Restricted additivity : RA)

Our main result in this section is the following theorem.

Theorem 20.2. N is the family of all 2-person bargaining solutions on B with the properties IR, PO, STI, PCO, and RA.

Comparing this result with Corollary 18.9, we have weakened SA to RA, strengthened WPO and HOM to PO and STI, and added PCO. Theorem 20.2 follows immediately from the following two propositions.

Proposition 20.3. Every Nash solution satisfies IR, PO, STI, PCO, and RA.

Proposition 20.4. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying IR, PO, STI, PCO, and RA. Let $t \in [0,1]$ be such that $\phi(\Delta) = (t,1-t)$.
Then $\phi = N^t$.
(Recall from Remark 12.10 that $\Delta = \text{comv}\{(1,0),(0,1)\} \in B$.)

In the proof of Proposition 20.3, we use the following lemma, which characterizes Pareto optimality of sums of points with the aid of parallel supporting lines.

Lemma 20.5. Let $S, T \in B$ and $z = x + y \in P(S + T)$ where $x \in S$, $y \in T$. Then we have
(i) $x \in P(S)$, $y \in P(T)$,
(ii) if ℓ is a supporting line of $S + T$ at z, then there exist supporting lines ℓ' and ℓ'' of S and T at x and y respectively, such that ℓ, ℓ' and ℓ'' are parallel,
(iii) if S and T are smooth at x and y respectively, then ℓ, ℓ' and ℓ'' in (ii) are unique (and $S + T$ is smooth at z).
Proof. (i) is straightforward by the definition of a Pareto optimal subset, and (iii) by (ii). To prove (ii), let ℓ be such a supporting line with a normal vector λ, then $\lambda \geq 0$, and $\lambda.z = \max\{\lambda.(s+t) : s \in S, t \in T\} = \max\{\lambda.s : s \in S\} + \max\{\lambda.t : t \in T\}$, hence $\lambda.x = \max\{\lambda.s : s \in S\}$ and $\lambda.y = \max\{\lambda.t : t \in T\}$, from which (ii) follows immediately. ∎

Proof of Proposition 20.3. Let $\phi \in N$. We only show that ϕ satisfies RA (for PCO see Theorem 19.6). First, let $\phi = N^t$ for some $t \in (0,1)$. Let $S, T \in B$ such that S and T are smooth at $x := N^t(S)$ and $y := N^t(T)$ respectively, and $x + y \in P(S + T)$. From Lemma 20.5 (iii) it follows that there exists a vector $\lambda \geq 0$, unique up to multiplication with a positive scalar, such that $\lambda.x = \max\{\lambda.s : s \in S\}$, $\lambda.y = \max\{\lambda.t : t \in T\}$, $\lambda.(x+y) = \max\{\lambda.v : v \in S + T\}$. From Lemma 12.1, it follows that $x = \gamma y$ for some $\gamma > 0$, hence $x + y = (1 + \gamma)y$. By applying Lemma 12.1 again, we obtain $N^t(S + T) = x + y$.
Secondly, let $\phi = D^1$, and S and T in B such that S and T are smooth at $D^1(S)$ and $D^1(T)$, respectively, and $D^1(S) + D^1(T) \in P(S + T)$. If $D^1_2(S) = D^1_2(T) = 0$, then $D^1_2(S) + D^1_2(T) = 0$, and so $D^1(S + T) = D^1(S) + D^1(T)$ since $D^1(S) + D^1(T) \in P(S)$. Otherwise, in view of Lemma 20.5 (iii), the unique supporting lines of S, T, and S + T, at $D^1(S)$, $D^1(T)$ and $D^1(S) + D^1(T)$ are the straight lines with equations $x_1 = D^1_1(S)$, $x_1 = D^1_1(T)$, and $x_1 = D^1_1(S + T)$, respectively. So $D^1(S + T) = D^1(S) + D^1(T)$ since $D^1(S) + D^1(T) \in P(S + T)$. The third case, $\phi = D^2$, is similar to the second one. ■

In the proof of Proposition 20.4 we will use the following two lemmas.

Lemma 20.6. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying IR, PO, and PCO. Let $S \in B$ such that S is smooth everywhere (i.e. at every point of P(S)) and such that the line of support of S at $\phi(S)$ has a normal vector with one coordinate equal to 0. Let $z \in P(S)$, $z \neq \phi(S)$. Then there exists an everywhere smooth $S' \in B$ with $S' \subset S$ and $z \in S'$ such that $\phi(S') \neq z$ and such that the line of support of S' at $\phi(S')$ has a strictly positive normal vector.
Proof. First note that $\phi(S) = D^1(S)$ or $\phi(S) = D^2(S)$. Assume $\phi(S) = D^2(S)$ (the other case is similar). If $\phi_1(S) = 0$, then an S' as in the lemma can easily be found by cutting off a suitable neighbourhood of $\phi(S)$ in S in a smooth way. Suppose now, that $\phi_1(S) > 0$.
First, choose $\bar{x} \in P(S)$ with $\phi_2(S) > \bar{x}_2 > z_2$ and such that $\phi_2(T) > z_2$ where T consists of all points of S except those strictly above the straight line through $\bar{x}$ and $\phi(S)$. Such a point $\bar{x}$ exists in view of PCO. The proof is complete if $\phi(T) \neq \phi(S)$ for then we can take, for S', the game T smoothed off at $\phi(S)$ and $\bar{x}$, in view of PCO. Now suppose $\phi(T) = \phi(S)$. For every ε with $0 \leqq \varepsilon \leqq \phi_1(S)$, let $S^\varepsilon \in B$ be the game consisting of all points of S except

those strictly above the straight line through $\bar{x}$ and the point $(\phi_1(S)-\varepsilon,\phi_2(S))$. Note that $S^0 = T$, so $\phi(S^0) = \phi(T) = \phi(S) = D^2(S) = D^2(S^0)$. Now let $\bar{\varepsilon} := \sup\{\varepsilon \in [0,\phi_1(S)] : \phi(S^\varepsilon) = D^2(S^\varepsilon)\}$. By PCO, $\phi(S^{\bar{\varepsilon}}) = D^2(S^{\bar{\varepsilon}})$. If $\bar{\varepsilon} = \phi_1(S)$, then we are back in the case of the first paragraph of the proof (where we assumed $\phi_1(S) = 0$). Otherwise, $0 \leqq \bar{\varepsilon} < \phi_1(S)$. Then take η with $\bar{\varepsilon} < \eta < \phi_1(S)$ small enough to have $(D_2^2(S^\eta) >) \ \phi_2(S^\eta) > z_2$. And take for S' the game S^η smoothed off at $D^2(S^\eta)$ and $\bar{x}$. ■

Lemma 20.7. Let $\phi : B \to \mathbb{R}^2$ be a 2-person bargaining solution with the properties IR, PO, STI, PCO, and RA. Let $\mu \in N$ be such that $\phi(\Delta) = \mu(\Delta)$. Let $T \in B$ be such that $P(T) \supset \operatorname{conv}\{v,w\}$ where $v, w \in \mathbb{R}^2$ satisfy $v_1 + v_2 = w_1 + w_2 = \alpha > 0$, $v_2 < 0$, $w_1 < 0$. Then $\phi(T) = \mu(T)$.

Proof. (See Fig. 20.1.) By STI, $\phi(\delta\Delta) = \mu(\delta\Delta)$ for every $\delta \in (0,\infty)$. Fix $\delta \in (0,\alpha)$. Fix $0 < \varepsilon < \min\{v_1-\delta, w_2-\delta, -v_2, -w_1\}$.
Let $D \in B$ be given by the following constraints :

$\{x \in W(D) : x_1 \leqq 0\} = \{(x_1 + \varepsilon, x_2 - \delta - \varepsilon) : x \in W(T), x_1 \leqq -\varepsilon\}$,

$\{x \in W(D) : x \geqq 0\} = \{x \geqq 0 : x_1 + x_2 = \alpha - \delta\}$,

$\{x \in W(D) : x_2 \leqq 0\} = \{(x_1 - \delta - \varepsilon, x_2 + \varepsilon) : x \in W(T), x_2 \leqq -\varepsilon\}$.

Let $E \in B$ be given by $E := \operatorname{comv}\{(\delta+\varepsilon,-\varepsilon),(-\varepsilon,\delta+\varepsilon)\}$. Then $E + D = T$. Note that E and D are smooth at every $x \in P(E) \cap \mathbb{R}^2_+$ and $y \in P(D) \cap \mathbb{R}^2_+$, and that all supporting lines at these points are parallel, with a normal vector $\lambda = (1,1)$. In particular, $x + y \in P(T)$ for every $x \in P(E) \cap \mathbb{R}^2_+$, $y \in P(D) \cap \mathbb{R}^2_+$. So by PO, IR, and RA, $\phi(T) = \phi(D) + \phi(E)$, hence $\phi_1(E) \leqq \phi_1(T) \leqq \phi_1(E) + \alpha - \delta$ and $\phi_2(E) \leqq \phi_2(T) \leqq \phi_2(E) + \alpha - \delta$. Letting now ε approach 0 gives, by PCO and the fact that $\phi(\delta\Delta) = \mu(\delta\Delta)$,

$\mu_1(\delta\Delta) \leqq \phi_1(T) \leqq \mu_1(\delta\Delta) + \alpha - \delta$, $\mu_2(\delta\Delta) \leqq \phi_2(T) \leqq \mu_2(\delta\Delta) + \alpha - \delta$.

Letting δ approach α, we obtain $\phi(T) = \mu(\alpha\Delta)$; hence, since by definition of μ : $\mu(\alpha\Delta) = \mu(T)$, we have $\phi(T) = \mu(T)$. ■

Proof of Proposition 20.4. (See Fig. 20.2.) Let $\mu \in N$ be the solution such that $\mu(\Delta) = \phi(\Delta) = (t,1-t)$. Suppose there exists an $S \in B$ such that

(20.1) $\phi(S) \neq \mu(S)$.

By PCO of ϕ and μ we may suppose that S is smooth everywhere, and by Lemma 20.6 (with $\mu(S)$ in the role of z), that the supporting line of S at $\phi(S)$ has a strictly positive normal vector λ. By STI, we may further suppose that $\lambda = (1,1)$ and $\phi_1(S) + \phi_2(S) = 1$. Then we have, by Lemma 12.1 and (20.1),

(20.2) $\phi(S) \neq (t,1-t)$.

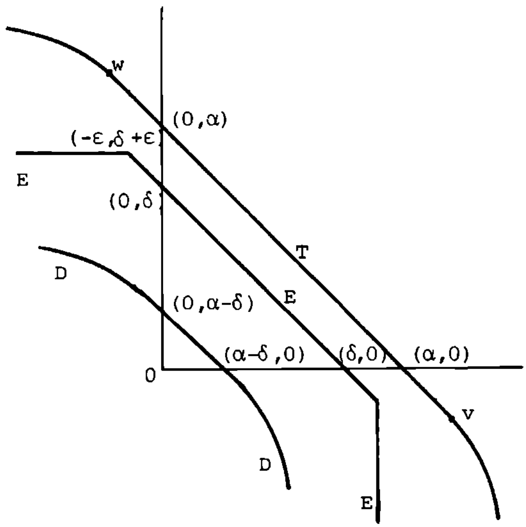


Figure 20.1.

Let $T := \operatorname{comv}\{(3,-2),(-2,3)\}$, then, by Lemma 20.7 and $\mu(T) = (t,1-t)$, we have

(20.3) $\phi(T) = (t,1-t)$.

Further, STI and Lemma 20.7 applied to $S + T$, give

(20.4) $\phi(S + T) = 2(t,1-t)$.

On the other hand, since S is smooth at $\phi(S)$, T is smooth at $(t,1-t)$, and $\phi(S) + (t,1-t) \in P(S + T)$, we have by RA and (20.3)

(20.5) $\phi(S + T) = \phi(S) + (t,1-t)$.

By combining (20.4) and (20.5) we obtain $\phi(S) = (t,1-t)$, in contradiction with (20.2). Hence (20.1) must be false, so $\phi(S) = \mu(S)$ for all $S \in B$. ∎

With the proof of Proposition 20.4, also the proof of Theorem 20.2 is completed. The following example shows that we cannot dispense with the PCO property in this theorem.

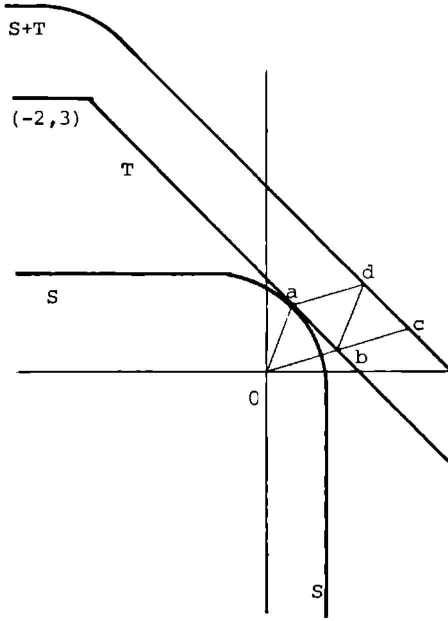


Figure 20.2.

a = $\phi(S)$
b = $(t,1-t)$
c = $2(t,1-t)$
d = $(t,1-t)+\phi(S)$

Example 20.8. We construct a solution $\phi : B \to \mathbb{R}^2$ by first defining it for all games S with $h(S) = (1,1)$. By applying the appropriate scale transformations, the definition is then extended to B, which guarantees that ϕ satisfies STI. So let $S \in B$ with $h(S) = (1,1)$. We define $\phi(S)$ as follows. If S is smooth at N(S), then $\phi(S) := N(S)$. If S is not smooth at N(S), then also $\phi(S) := N(S)$ except for the case that there exists exactly one other point $x \in P(S) \cap \mathbb{R}^2_+$ such that S is not smooth at x; in that case, $\phi(S) := x$. It is straightforward to verify that this ϕ, besides STI, satisfies IR, PO, and RA, but not PCO.

The smoothness condition in the definition of restrictive additivity cannot be dispensed with, either.

Example 20.9. Let $S := \text{com}\{(2,1)\}$. Then $N(\Delta + S) = (2,2) \neq (\frac{1}{2},\frac{1}{2}) + (2,1) = N(\Delta) + N(S) \in P(\Delta + S)$. Here Δ is smooth at $N(\Delta)$, but S is not smooth at N(S).

We conclude this section with two remarks.

Remark 20.10. We have the following link between the results in this section and those in section 18. For a solution $\phi : B \to \mathbb{R}^2$ and $S, T \in B$, say that

RA *applies to* ϕ, S *and* T if S and T are smooth at $\phi(S)$ and $\phi(T)$ respectively, and $\phi(S) + \phi(T) \in P(S + T)$. Then, as a consequence of Lemmas 12.1 and 20.5, for every $t \in (0,1)$, if RA applies to N^t, S and T, then $N^t(S) = E^p(S)$, $N^t(T) = E^p(T)$, and $N^t(S + T) = E^p(S + T)$, for some $p > 0$.

Remark 20.11. The main result of this section is related to the main result in Aumann (1985), where an axiomatic characterization of the so-called Non-Transferable Utility value is given (cf. Shapley (1969)). Aumann uses a Conditional Additivity axiom, which is stronger than restricted additivity, in that it does not require smoothness. However, Aumann restricts attention to smooth games, where we have the Pareto continuity property to take care of non-smoothness.

CHAPTER 7

MULTISOLUTIONS AND PROBABILISTIC SOLUTIONS

We restrict attention again to 2-person bargaining games. In section 21 we consider so-called multisolutions which assign to each bargaining game not one outcome but a set of outcomes. Section 21 generalizes results of sections 11 - on IIA-solutions - and 15 (on IM-solutions); it is based on Peters, Tijs and de Koster (1983).

In section 22 we introduce probabilistic solutions which assign probabilities to subsets of outcomes of a bargaining game. In particular, the results of section 11 on IIA-solutions are generalized again, and there is a close link with the results of section 21 on IIA-multisolutions. Section 22 is based on Peters and Tijs (1983).

21. MULTISOLUTIONS

A (2-person) bargaining solution $\phi : B \to \mathbb{R}^2$ assigns one point of S to each $S \in B$. Recall that, for nonempty sets X and Y, a *multifunction* $F : X \to Y$ is a map assigning to each element of X a nonempty subset of Y; in particular, we have the following definition.

Definition 21.1. A *multisolution* $\phi : B \to \mathbb{R}^2$ is a multifunction with $\phi(S) \subset S$ for every $S \in B$. We call ϕ *closed-valued* if $\phi(S)$ is closed for every $S \in B$.

With every bargaining solution $\phi : B \to \mathbb{R}^2$ we can associate a multisolution $\tilde{\phi} : B \to \mathbb{R}^2$ by $\tilde{\phi}(S) := \{\phi(S)\}$ for every $S \in B$. We will write ϕ instead of $\tilde{\phi}$. Also, we will often omit the braces in case of one-point sets.

The purpose of this section is to characterize multisolutions with the aid of generalizations of properties of bargaining solutions. We give a list of the properties which play a role in this section, for a multisolution $\phi : B \to \mathbb{R}^2$.

Definition 21.2.

(i) *Individual rationality* (IR) : $\phi(S) \subset \mathbb{R}^2_+$ for every $S \in B$.

(ii) *[Weak] Pareto optimality* ([W]PO) : $\phi(S) \subset P(S)$ $[W(S)]$ for every $S \in B$.

(iii) *Scale transformation invariance* (STI) : $\phi(aS) = a\phi(S)$ for every scale transformation $a \in \mathbb{R}^2_{++}$ and every $S \in B$.

(iv) *Independence of irrelevant alternatives* (IIA) : $\phi(S) = \phi(T) \cap S$ for all S and T in B with $S \subset T$ and $\phi(T) \cap S \neq \emptyset$.

(v) *Restricted monotonicity* (RM) : $\phi(S) \subset \phi(T) - \mathbb{R}^2_+$ and $\phi(T) \subset \phi(S) + \mathbb{R}^2_+$ for all S and T in B with $S \subset T$ and $h(S) = h(T)$.
(For X, $Y \subset \mathbb{R}^n$, by X-Y we denote $\{x-y : x \in X, y \in Y\}$. For h(S), see Def. 9.10.)

Note that for bargaining solutions all these properties coincide with the existing properties with the same names, which justifies the use of these same names.

The IIA-property as formulated here, can also be found in Kaneko (1980). Aumann (1985) proposes the following "IIA"-property for a multisolution $\phi : B \to \mathbb{R}^2$: for all S and T in B with $S \subset T$ and $\phi(T) \cap S \neq \emptyset$, $\phi(S) \supset \phi(T) \cap S$. For bargaining solutions, both versions coincide. Note however, that the multisolution given by $S \mapsto P(S)$ for every $S \in B$, satisfies Aumann's "IIA" but not our IIA : "IIA" is strictly weaker than IIA.

The remainder of this section consists of two parts. In the first part, part A, we shall describe the family of all multisolutions satisfying IR, WPO, STI and IIA. Notice that, apart from considering multisolutions instead of bargaining solutions, we also generalize the results of section 11 by replacing PO by WPO. In part B we describe the family of all closed-valued multisolutions satisfying PO, STI, and RM. (We use RM instead of a multisolution version of IM merely for convenience.) Thus, the second part provides a generalization of especially Propositions 15.13 and 15.14.

Part A : Independence of irrelevant alternatives

We start with a few notations.

Definition 21.3. For $S \in B$, $\bar{p}(S)$ and $\underline{p}(S)$ were introduced in Definition 11.2. By $\bar{w}(S)$, we denote the point in W(S) with first coordinate 0, and by $\underline{w}(S)$ the point in W(S) with second coordinate 0. By $\bar{W}(S)$, we denote $\mathrm{conv}\{\bar{w}(S), \bar{p}(S)\}$, and by $\underline{W}(S)$ we denote $\mathrm{conv}\{\underline{w}(S),\underline{p}(S)\}$. Finally, by $\bar{M}(S)$ we denote $(\bar{W}(S)\setminus\{\bar{w}(S)\}) \cup \{\bar{p}(S)\}$, and by $\underline{M}(S)$ the set $(\underline{W}(S)\setminus\{\underline{w}(S)\}) \cup \{\underline{p}(S)\}$.

Note that $\underline{w},\bar{w},\underline{p},\bar{p} : B \to \mathbb{R}^2$ are bargaining solutions with, of course, $\underline{p} = D^1 = N^1$ and $\bar{p} = D^2 = N^0$, and that $\underline{w},\bar{w},\underline{p},\bar{p},\underline{W},\bar{W},\underline{M},\bar{M} : B \to \mathbb{R}^2$ are multisolutions.

Our purpose in part A is, to prove the following theorem.

<u>Theorem 21.4.</u> $N \cup \{\bar{w},\underline{w}\} \cup \{\bar{W},\underline{W},\bar{M},\underline{M}\}$ is the family of all multisolutions : $B \to \mathbb{R}^2$ which have the properties IR, WPO, STI, and IIA.

We leave it to the reader to verify that all the mentioned multisolutions satisfy the mentioned properties. The other part of the theorem will be proved with the aid of a string of lemmas. Another notation : $\square := \mathrm{com}\{(1,1)\}$.

<u>Remark 21.5.</u> In Peters, Tijs, and de Koster (1983), compactness instead of comprehensiveness of bargaining games is assumed. A consequence is that, there, four additional multisolutions are found to satisfy the four properties of Theorem 21.4. See the mentioned paper for the details.

<u>Lemma 21.6.</u> Let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying IR, STI, and IIA. Then :

(i) If $\phi(\square) = (0,1)$, then $\phi = \bar{w}$.

(ii) If $\phi(\square) = (1,0)$, then $\phi = \underline{w}$.

<u>Proof.</u> We only show (i). Let $\phi(\square) = (0,1)$ and $S \in B$, then by STI : $\phi(h(S)\square) = \bar{w}(S)$. So by IIA applied to $S_+ \subset h(S)\square$, we have $\phi(S_+) = \bar{w}(S)$. Finally, by IR and IIA applied to $S_+ \subset S$, we obtain $\phi(S) = \bar{w}(S)$. ■

The following lemma can be proved in the same way as Lemma 21.6. Details are omitted.

<u>Lemma 21.7.</u> Let $\phi : B \to \mathbb{R}^2$ be a multisolution with the properties IR, STI, and IIA. Then :

(i) If $\phi(\square) = \bar{W}(\square)$, then $\phi = \bar{W}$.

(ii) If $\phi(\square) = \underline{W}(\square)$, then $\phi = \underline{W}$.

We need four more lemmas before we can prove Theorem 21.4.

Lemma 21.8. Let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying IR, WPO, STI, and IIA. Suppose $a,b \in \phi(S)$ with $a \neq b$, for some $S \in B$.
Then $a_1 = b_1$ or $a_2 = b_2$.

Proof. Suppose that $a_1 \neq b_1$ and $a_2 \neq b_2$, say $a_1 < b_1$ and $a_2 > b_2$ (noting that ϕ satisfies WPO). In view of IR, we distinguish two cases : (i) $a_1 = 0$, (ii) $a_1 > 0$. In case (i), let $T := \mathrm{comv}\{a,b\}$. Then $(\frac{1}{2},1)T \subset T$, and $a \in (\frac{1}{2},1)T \cap \phi(T) \neq \emptyset$ since $\phi(T) = \phi(S) \cap T \supset \{a,b\}$ by IIA; so, also by IIA, $\phi((\frac{1}{2},1)T) = \phi(T) \cap (\frac{1}{2},1)T$. By STI, $(\frac{1}{2}b_1,b_2) \in \phi((\frac{1}{2},1)T)$, hence $(\frac{1}{2}b_1,b_2) \in \phi(T)$, in contradiction with WPO. So in case (i) : $a_1 = b_1$ or $a_2 = b_2$.
For case (ii), let $c := (a_1(\frac{1}{2}a_1 + \frac{1}{2}b_1)^{-1}, a_2(\frac{1}{2}a_2 + \frac{1}{2}b_2)^{-1}) \in \mathbb{R}^2_{++}$, T as above, and $E := \mathrm{comv}\{b,\frac{1}{2}(a+b)\}$. (See Fig. 21.1.) Then an elementary calculation shows that $cE \subset T$ and $cb \notin W(T)$. Since $a = c(\frac{1}{2}(a+b)) \in cE \cap \phi(T)$, by IIA : $\phi(cE) = \phi(T) \cap cE$. Since, by IIA, $b \in \phi(E)$, by STI : $cb \in \phi(cE)$. So $cb \in \phi(T)$, a contradiction since $cb \notin W(T)$. Hence also in case (ii) : $a_1 = b_1$ or $a_2 = b_2$. ■

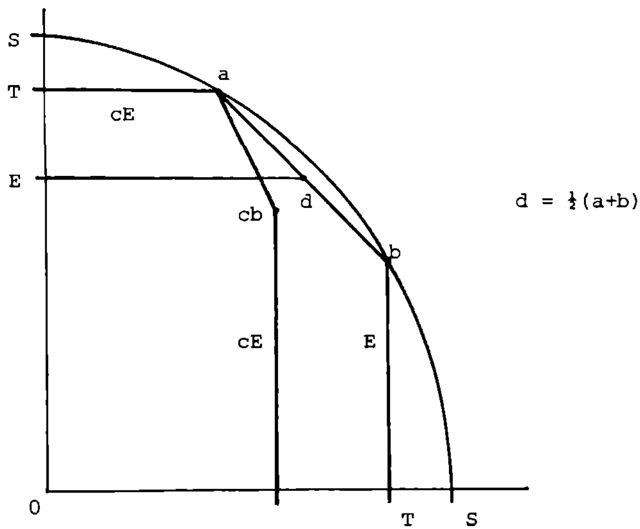


Figure 21.1.

Lemma 21.9. Let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying IR, WPO, STI, and IIA. Then :

(i) If $(\alpha,1) \in \phi(\square)$ for some $\alpha \in (0,1)$, and $\phi \neq \bar{W}$, then $\phi = \bar{M}$.

(ii) If $(1,\alpha) \in \phi(\square)$ for some $\alpha \in (0,1)$, and $\phi \neq \underline{W}$, then $\phi = \underline{M}$.

Proof. We only show (i). Let $(\alpha,1) \in \phi(\square)$ for some $\alpha \in (0,1)$, and $\phi \neq \bar{W}$. Then, by IIA, $(\alpha,1) \in \phi(\text{comv}\{(\alpha,1)\}) = \phi((\alpha,1)^{\square})$, so by STI, $(1,1) \in \phi(\square)$. Let $\alpha \leqq \beta \leqq 1$, then, since $(\alpha,1) \in \phi(\square) \cap (\beta,1)^{\square}$, we have by IIA : $\phi((\beta,1)^{\square}) = (\beta,1)^{\square} \cap \phi(\square)$, so, because $(\beta,1) \in \phi((\beta,1)^{\square})$ by STI and $(1,1) \in \phi(\square)$, we have $(\beta,1) \in \phi(\square)$. We have shown :

(21.1) $(\beta,1) \in \phi(\square)$ for all $\beta \in [\alpha,1]$.

By (21.1) and IIA applied to $(\alpha,1)^{\square} \subset \square$, we obtain $\phi((\alpha,1)^{\square}) = (\alpha,1)^{\square} \cap \phi(\square)$, hence by (21.1) and STI : $(\beta,1) \in \phi(\square)$ for all $\beta \in [\alpha^2,1]$. So, since $\lim_{n\to\infty} \alpha^n = 0$, we have $\bar{M}(\square) \subset \phi(\square)$. By Lemma 21.8, IR, and Lemma 21.7(i) we then have : $\phi(\square) = \bar{M}(\square)$.

Let $S \in B$. If $\bar{p}_1(S) > 0$, then $\phi(S) = \bar{M}(S)$ by a proof analogous to the proof of Lemma 21.6. Otherwise, let $x \in W(S)$, $x \geqq 0$, $x_1 > 0$. Take $y \in S$ such that $0 < y_1 < x_1$ and $y_2 > x_2$. Then $x \notin \bar{M}(\text{comv}\{x,y\}) = \phi(\text{comv}\{x,y\})$, so by IIA applied to $\text{comv}\{x,y\} \subset S$, we have $x \notin \phi(S)$. So $\phi(S) = \{\bar{p}(S)\} = \bar{M}(S)$, also in this case. ■

Lemma 21.10. Let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying IR, WPO,STI, and IIA. Then :

(i) If $(0,1),(1,1) \in \phi(\square)$, then $\phi = \bar{W}$. (ii) If $(1,0),(1,1) \in \phi(\square)$, then $\phi = \underline{W}$.

Proof. We only show (i). Suppose $(0,1),(1,1) \in \phi(\square)$. Take $0 < \beta < 1$. By IIA : $\phi((\beta,1)^{\square}) = \phi(\square) \cap (\beta,1)^{\square}$, hence $(\beta,1) \in \phi(\square)$, since $(\beta,1) \in \phi((\beta,1)^{\square})$ by STI. Since β was arbitrary, we have $\phi(\square) = \bar{W}(\square)$, and hence $\phi = \bar{W}$, by Lemma 21.7 (i). ■

Lemma 21.11. Let $\phi : B \to \mathbb{R}^2$ be multisolution satisfying IR, WPO, STI, and IIA. If $\phi(\square) = (1,1)$, then $\phi \in N$.

Proof. Assume $\phi(\square) = (1,1)$. Suppose $\phi(S) \setminus P(S) \neq \emptyset$ for some $S \in B$. Say $x \in \phi(S) \setminus P(S)$, then $y \in W(S)$ exists with $y \geqq x$, $y \neq x$. By STI : $\phi(y^{\square}) = Y$. By IIA : $x \in \phi(y^{\square})$. So we have a contradiction from which we conclude that ϕ satisfies PO. Hence, in view of Lemma 21.8, ϕ is a bargaining solution. Now the proof is complete by Theorem 11.8. ■

Proof of Theorem 21.4. As noted before, the proof of the fact that the mentioned multisolutions satisfy the mentioned properties, is left to the reader. Now let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying IR, WPO, STI, and IIA. We distinguish a few cases.

(i) $\phi(\square) = (1,1)$. Then $\phi \in N$ by Lemma 21.11.

(ii) $\phi(\square) \cap \bar{W}(\square)\setminus\{(1,1)\} \neq \emptyset$. Then, by Lemma 21.8, $\phi(\square) \subset \bar{W}(\square)$. If $\phi(\square) = \bar{W}(\square)$, then $\phi = \bar{W}$ by Lemma 21.7 (i). If $\phi \neq \bar{W}$ and $(\alpha,1) \in \phi(\square)$ for some $\alpha \in (0,1)$, then $\phi = \bar{M}$ by Lemma 21.9 (i). Otherwise $\phi(\square) \subset \{(0,1),(1,1)\}$, hence $\phi(\square) = (0,1)$ by Lemma 21.10 (i), so $\phi = \bar{w}$ by Lemma 21.6 (i).

(iii) $\phi(\square) \cap \underline{W}(\square)\setminus\{(1,1)\} \neq \emptyset$. Analogous to (ii), left to the reader. ■

Part B : Restricted monotonicity

We start by introducing an extension of the concept of a monotonic curve of section 15.

Definition 21.12. A *monotonic multicurve* is a multifunction $\mu : [1,2] \to \nabla$, where $\nabla := \text{conv}\{(1,1),(1,0),(0,1)\}$, with the following properties :

(21.2) For every $t \in [1,2]$, $\mu(t)$ is a non-empty closed subset of $\{x \in \nabla : x_1 + x_2 = t\}$.

(21.3) For all $s, t \in [1,2]$ with $s \leq t$: $\mu(t) \subset \mu(s) + \mathbb{R}^2_+$, $\mu(s) \subset \mu(t) - \mathbb{R}^2_+$.

The family of all monotonic multicurves is denoted by M.

We will see that to monotonic multicurves correspond closed RM-multisolutions. We first take a closer look at monotonic multicurves. For $\mu \in M$, let $D(\mu) := \bigcup_{t \in [1,2]} \mu(t)$. Further, recall that a multifunction $F : X \to Y$ (with X and Y topological spaces) is called *upper semicontinuous* if for every open $U \subset Y$ the set $F^+(U) := \{x \in X : F(x) \subset U\}$ is open in X; and *lower semicontinuous* if for every open $U \subset Y$ the set $F^-(U) := \{x \in X : F(x) \cap U \neq \emptyset\}$ is open in X.

Lemma 21.13. Let $\mu \in M$. Then μ is upper and lower semicontinuous and $D(\mu)$ is a closed subset of ∇.

Proof. (i) First we prove that μ is upper semicontinuous. Let U be an open subset of ∇ (where ∇ is provided with the relative topology). We show that $\mu^+(U)$ is open in $[1,2]$. This is true if $\mu^+(U) = \emptyset$. Suppose $t^\circ \in \mu^+(U)$. Then $\mu(t^\circ)$ is a compact subset of U. So we can take an $\varepsilon > 0$ such that $T := \{x \in \nabla : \|x-y\|_1 < \varepsilon \text{ for some } y \in \mu(t^\circ)\} \subset U$. Let $t \in [1,2]$, with $t^\circ - \varepsilon < t \leqq t^\circ$. By (21.3), for each $x \in \mu(t)$ there is an $y \in \mu(t^\circ)$ with $x \leqq y$. Then (by (21.2)) $\|y-x\|_1 = (y_1-x_1) + (y_2-x_2) = (y_1+y_2) - (x_1+x_2) = t^\circ - t < \varepsilon$, so $x \in T \subset U$. Hence $\mu(t) \subset U$, $t \in \mu^+(U)$. If $t \in [1,2]$ with $t^\circ \leqq t < t^\circ + \varepsilon$, then there is, for each $x \in \mu(t)$, a $z \in \mu(t^\circ)$ with $z \leq x$. We then find, similarly, $\mu(t) \subset U$, $t \in \mu^+(U)$.

(ii) Next, we prove that μ is lower semicontinuous. Let U again be an open subset of ∇. We show that $\mu^-(U)$ is open in $[1,2]$. Suppose $t^1 \in \mu^-(U)$ and $x \in \mu(t^1) \cap U$. There is a $\delta > 0$ with $\{y \in \nabla : \|y - x\|_1 < \delta\} \subset U$. Let $s \in [t^1, t^1+\delta) \cap [1,2]$. Then $u \geqq x$ for some $u \in \mu(s)$. Then $\|u - x\|_1 < \delta$, so $u \in U$, $s \in \mu^-(U)$. Similarly for $s \in (t^1-\delta, t^1] \cap [1,2]$.

(iii) Finally, we show that $D(\mu)$ is a closed subset of ∇. Let $H(\mu) := \{(t,x) : t \in [1,2],\ x \in \mu(t)\}$ be the graph of μ. Then, since μ is upper semicontinuous, $H(\mu)$ is a closed subset of $[1,2] \times \nabla$ (see Hildenbrand and Kirman (1976, p.194)). So $H(\mu)$ is compact, and also $D(\mu) = \pi(H(\mu))$ is compact where $\pi : [1,2] \times \nabla \to \nabla$ is the continuous function with $\pi(t,x) = x$ for all $(t,x) \in [1,2] \times \nabla$. ∎

Lemma 21.14. Let $S \in \mathcal{B}$ with $h(S) = (1,1)$, and $\mu \in \mathcal{M}$.

(i) If $a \in D(\mu)$ and $(a - \mathbb{R}^2_+) \cap P(S) \neq \emptyset$, then $(a - \mathbb{R}^2_+) \cap P(S) \cap D(\mu) \neq \emptyset$.

(ii) If $b \in D(\mu)$ and $(b + \mathbb{R}^2_+) \cap P(S) \neq \emptyset$, then $(b + \mathbb{R}^2_+) \cap P(S) \cap D(\mu) \neq \emptyset$.

Proof. We only prove (ii). If $b \in P(S)$ or $(1,1) \in P(S)$, then there is nothing to prove. So, suppose $b \notin P(S)$ and $(1,1) \notin P(S)$. Let $K := \{x \in \nabla : b \leqq x \leqq (1,1)\}$ and let $\beta := b_1 + b_2$. Let $\bar{\mu} : [\beta,2] \to \nabla$ be the multifunction with $\bar{\mu}(s) = \mu(s) \cap K$ for all $s \in [\beta,2]$. In view of (21.3), $\bar{\mu}(s) \neq \emptyset$ for each $s \in [\beta,2]$, and $\bar{\mu}$ is upper and lower semicontinuous in view of Lemma 21.13 and the fact that $\bar{\mu}$ is the restriction to $[\beta,2]$ of a monotonic multicurve. Now let

$V := \{x \in K : x \notin P(S),\ (x - \mathbb{R}^2_+) \cap P(S) \neq \emptyset\}$, $I_1 := \{t \in [\beta,2] : \bar{\mu}(t) \subset V\}$,

$W := \{x \in K : x \notin P(S),\ (x + \mathbb{R}^2_+) \cap P(S) \neq \emptyset\}$, $I_2 := \{t \in [\beta,2] : \bar{\mu}(t) \cap W \neq \emptyset\}$.

Note that $2 \in I_1$, that $\beta \in I_2$ because $b \in \bar{\mu}(\beta)$, and that $I_1 \cap I_2 = \emptyset$.

Since V and W are open subsets of K (in the relative topology) it follows from the upper and lower semicontinuity of the multifunction $\bar{\mu}$, that I_1 and I_2 are open subsets of $[\beta,2]$. Now $I_1 \cup I_2 = [\beta,2]$ if $(b+\mathbb{R}^2_+) \cap P(S) \cap D(\mu) = \emptyset$ and that is in contradiction with the connectedness of $[\beta,2]$. Hence $(b+\mathbb{R}^2_+) \cap P(S) \cap D(\mu) \neq \emptyset$. ∎

We now associate with each $\mu \in M$ a multifunction $\pi^\mu : B \to \mathbb{R}^2$. Let $S \in B$. If $h(S) = (1,1)$ then let $\pi^\mu(S) := D(\mu) \cap P(S)$. In general, let $\pi^\mu(S) := h(S)\pi^\mu((h_1(S)^{-1}, h_2(S)^{-1})S)$.

<u>Proposition 21.15.</u> Let $\mu \in M$. Then π^μ is a closed-valued multisolution satisfying PO, STI, and RM.

<u>Proof.</u> By Lemma 21.14, $\pi^\mu(S) \neq \emptyset$ for each $S \in B$. Since $D(\mu)$ is closed in view of Lemma 21.13 and also $P(S)$ is closed, we have $\pi^\mu(S)$ closed for each $S \in B$. Further, it is obvious that π^μ satisfies PO and STI. To prove that π^μ satisfies the RM-property, let S and T in B with $h(S) = h(T) = (1,1)$ and $S \subset T$. Take $a \in \pi^\mu(T)$. Then $(a-\mathbb{R}^2_+) \cap P(S) \neq \emptyset$. By Lemma 21.14 (i) : $\emptyset \neq (a-\mathbb{R}^2_+) \cap P(S) \cap D(\mu) = (a-\mathbb{R}^2_+) \cap \pi^\mu(S)$. This implies that $\pi^\mu(T) \subset \pi^\mu(S) + \mathbb{R}^2_+$. Analogously, it follows with Lemma 21.14 (ii) that $\pi^\mu(S) \subset \pi^\mu(T) - \mathbb{R}^2_+$. ∎

<u>Lemma 21.16.</u> Let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying PO and RM. Let $S, T \in B$ with $h(S) = h(T)$. Then $\phi(S) \cap P(T) \subset \phi(T)$.

<u>Proof.</u> Let $D := S \cap T \in B$. Then $h(D) = h(S) = h(T)$. Take $y \in \phi(S) \cap P(T)$. Then $y \in P(D)$. By RM, $x \leqq y$ for some $x \in \phi(D)$.
Since also $x \in P(D)$ by PO, we have $x = y$, so $y \in \phi(D)$. By RM again, there is a $z \in \phi(T)$ with $y \leqq z$. Since $y, z \in P(T)$ we have $y = z \in \phi(T)$. Hence, $\phi(S) \cap P(T) \subset \phi(T)$. ∎

<u>Lemma 21.17.</u> Let $\phi : B \to \mathbb{R}^2$ be a multisolution satisfying PO and RM. Let $S \in B$. Then :
(i) $\phi(S) = \phi(S_+)$, (ii) ϕ is individually rational.

<u>Proof.</u> (ii) follows from (i) and PO, and (i) from PO and RM. ∎

<u>Proposition 21.18.</u> Let $\phi : B \to \mathbb{R}^2$ be a closed-valued multisolution satisfying PO, STI, and RM. Then $\phi = \pi^\mu$ for some $\mu \in M$.

Proof. Let $V(t) := \text{comv}\{(1,0),(1,t-1),(t-1,1),(0,1)\}$ for each $t \in [1,2]$. Define the multifunction $\mu : [1,2] \to \mathbb{R}^2$ by $\mu(t) := \phi(V(t))$ for all $t \in [1,2]$. Then $\mu(t)$ is a non-empty closed subset of $P(V(t)) = \{x \in \nabla : x_1 + x_2 = t\}$ and for $1 \leq s \leq t \leq 2$ we have by RM of ϕ :

$\mu(t) = \phi(V(t)) \subset \phi(V(s)) + \mathbb{R}^2_+ = \mu(s) + \mathbb{R}^2_+$,

$\mu(s) = \phi(V(s)) \subset \phi(V(t)) - \mathbb{R}^2_+ = \mu(t) - \mathbb{R}^2_+$.

Hence, $\mu \in M$. We want to show that $\phi = \pi^\mu$. In view of STI, it is sufficient to show that $\phi(S) = \pi^\mu(S)$ where $S \in B$ with $h(S) = (1,1)$. Note that $\phi(V(t)) = \pi^\mu(V(t))$ for all $t \in [1,2]$. Take $x \in \pi^\mu(S)$. Let $s := x_1 + x_2$. Then, by applying Lemma 21.16 we obtain :

$x \in \pi^\mu(S) \cap P(V(s)) \Rightarrow x \in \pi^\mu(V(s)) = \phi(V(s))$,

$x \in \phi(V(s)) \cap P(S) \Rightarrow x \in \phi(S)$.

Hence, $\pi^\mu(S) \subset \phi(S)$. For the converse, take an $y \in \phi(S)$ and let $t := y_1 + y_2$ (note that $y \in \nabla$ in view of PO of ϕ and Lemma 21.17(ii)). Then, by applying Lemma 21.16 again :

$y \in \phi(S) \cap P(V(t)) \Rightarrow y \in \phi(V(t)) = \pi^\mu(V(t))$,

$y \in \pi^\mu(V(t)) \cap P(S) \Rightarrow y \in \pi^\mu(S)$.

So, $\phi(S) \subset \pi^\mu(S)$. We have proved that $\phi(S) = \pi^\mu(S)$. ■

The main result of this part of the section follows from Propositions 21.15 and 21.18.

Theorem 21.19. $\{\pi^\mu : \mu \in M\}$ is the family of all closed-valued multisolutions : $B \to \mathbb{R}^2$ which satisfy PO, STI, and RM.

In Peters, Tijs, and de Koster (1983), it is shown that PO in Theorem 21.19 can be relaxed to WPO if a class of not necessarily comprehensive bargaining games is considered, instead of B. In the present Theorem 21.19 however, PO cannot be replaced by WPO :

Example 21.20. All following multisolutions $\phi : B \to \mathbb{R}^2$ satisfy WPO, STI, RM, but not PO. For every $S \in B$:

(i) $\phi(S) = W(S)$, (ii) $\phi(S) = W(S) \cap \mathbb{R}^2_+$,

(iii) $\phi(S) = h(S)(D(\mu) \cap W((h_1(S)^{-1}, h_2(S)^{-1})S))$ for a fixed arbitrary $\mu \in M$,

(iv) $\phi(S) = \{([\bar{p}_1(S)]^2, \bar{p}_2(S)), ([\bar{p}_1(S)]^3, \bar{p}_2(S))\}$ if $h(S) = (1,1)$ and $\phi(S) = h(S)\phi((h_1(S)^{-1}, h_2(S)^{-1})S)$ otherwise.

Whereas a bargaining solution assigns exactly one point to a bargaining game, and a multisolution a set of points, a so-called probabilistic solution assigns a probability distribution (measure) to each bargaining game. We shall see that (multi)solutions can be related to probabilistic solutions. We shall study "probabilistic" versions of the independence of irrelevant alternatives property, and characterize families of probabilistic solutions with the aid of such properties. Thereby, like in the previous section, we extend results of section 11. As everywhere in this chapter, we confine ourselves to the 2-person case.

For $S \in B$, we denote by $\sigma(S)$ the Borel σ-algebra of S. A *probability measure on* S is a map $\phi_S : \sigma(S) \to [0,1]$ such that $\phi_S(S) = 1$ and such that ϕ_S is σ-additive, i.e. $\phi_S(\bigcup_i E_i) = \sum_i \phi_S(E_i)$ if $E_1, E_2, \ldots$ is a sequence of pairwise disjoint elements in $\sigma(S)$. The *support of* ϕ_S, denoted $\mathrm{supp}(\phi_S)$, is defined by

$$\mathrm{supp}(\phi_S) := \{x \in S : \phi_S(E) \neq 0 \text{ for all } E \text{ in } \sigma(S) \text{ with } x \in E\}.$$

$M(S)$ denotes the family of all probability measures on S and $F(S) \subset M(S)$ the family of all probability measures with finite support.

A *probabilistic solution* is a map ϕ assigning to each $S \in B$ an element ϕ_S in $M(S)$. For $S \in B$ and $E \in \sigma(S)$, $\phi_S(E)$ can be interpreted as the probability that the final agreement between the players in the bargaining game S will be in E.

To a bargaining solution $\phi : B \to \mathbb{R}^2$ we associate a probabilistic solution $\tilde{\phi}$ by $\tilde{\phi}_S(\{\phi(S)\}) = 1$ for every $S \in B$. We call $\tilde{\phi}$ the *probabilistic solution corresponding to* ϕ, and will write ϕ instead of $\tilde{\phi}$. As in the previous section, we will often omit braces in case of one-point sets.

Further, we say that the (multi)solution $\tilde{\phi} : B \to \mathbb{R}^2$ *supports* the probabilistic solution ϕ if $\tilde{\phi}(S) = \mathrm{supp}(\phi_S)$ for every $S \in B$.

In the remainder of this section, we shall often use the abbreviation "p-solution" for "probabilistic solution". Many properties for bargaining solutions can be translated in an obvious way for a probabilistic solution ϕ :

Definition 22.1.

(i) *Individual rationality* (IR) : $\phi_S(S \cap \mathbb{R}^2_+) = 1$ for all $S \in B$.

(ii) [*weak*] *Pareto optimality* ([W]PO) : $\phi_S(P(S)) = 1$ [$\phi_S(W(S)) = 1$] for all

$S \in B$.

(iii) *Scale transformation invariance* (STI) : $\phi_{aS}(aE) = \phi_S(E)$ for all $S \in B$, $E \in \sigma(S)$, and $a \in \mathbb{R}^2_{++}$.

We propose the following "probabilistic" version of the IIA-property, for a p-solution ϕ. For solutions, this property coincides with the IIA-property given in Def. 11.1.

Definition 22.2. We call ϕ *independent of irrelevant alternatives* if, for all S and T in B with $S \subset T$, and every E in $\sigma(S)$, we have $\phi_S(E) \geq \phi_T(E)$. (Independence of irrelevant alternatives : IIA)

This IIA-property may be interpreted as follows. If the set of possible outcomes in a bargaining game is decreased, then every still available (Borel) subset of outcomes should have at least as large a probability of containing the final agreement of the game as it originally had. Two preliminary results with respect to this property are in order.

Lemma 22.3. Let ϕ be a p-solution satisfying IR and IIA. Then :

(i) $\phi_S(E) = \phi_{S_+}(E)$ for every $S \in B$ and $E \in \sigma(S_+)$.

(ii) For all $S, T \in B$ with $S_+ \subset T_+$, we have $\phi_S(E) \geq \phi_T(E)$ for every $E \in \sigma(S)$.

Proof. Let $S \in B$, $E \in \sigma(S_+)$. Then by IR and IIA : $\phi_{S_+}(E) = \phi_{S_+}(E \cap \mathbb{R}^2_+) \geq \phi_S(E \cap \mathbb{R}^2_+) = \phi_S(E)$ and $\phi_{S_+}((S \cap \mathbb{R}^2_+)\setminus E) \geq \phi_S((S \cap \mathbb{R}^2_+)\setminus E)$, hence $\phi_{S_+}(E) = \phi_S(E)$ since otherwise $\phi_{S_+}(S \cap \mathbb{R}^2_+) > 1$. This proves (i).
Let $T \in B$ with $S_+ \subset T_+$. Let $E' \in \sigma(S)$. Then $\phi_S(E') = \phi_S(E' \cap \mathbb{R}^2_+) = \phi_{S_+}(E' \cap \mathbb{R}^2_+) \geq \phi_{T_+}(E' \cap \mathbb{R}^2_+) = \phi_T(E' \cap \mathbb{R}^2_+) = \phi_T(E')$, by IR, IIA, and (i). This proves (ii). ∎

Lemma 22.4. Let $S \in B$ and let ϕ be a p-solution with the properties IR, STI, and IIA. Then $x \notin \mathbb{R}^2_+ \cap P(S) \cup \{0, \underline{w}(S), \tilde{w}(S)\}$ implies $\phi_S(x) = 0$.

Proof. Let $x \in S$. If $x \notin \mathbb{R}^2_+$, then $\phi_S(x) = 0$ by IR. Suppose $x \in \mathbb{R}^2_+$, and suppose there exists a set $S_x \subset S$ such that $y \geq x$ for all $y \in S_x$, S_x is countably infinite, and for every $y \in S_x$ there exists an $a \in \mathbb{R}^2_{++}$ with $ay = x$. By STI and IIA, if $y \in S_x$ and $ay = x$ for some $a \in \mathbb{R}^2_{++}$, then $\phi_S(y) = \phi_{aS}(x) \geq \phi_S(x)$ since $a \leq (1,1)$. If $\phi_S(x) > 0$, then summing for all $y \in S_x$

would yield $\infty = \phi_S(S_x)$, an impossibility. So $\phi_S(x) = 0$. The proof is complete by the observation that such a set S_x exists for every $x \in S \cap \mathbb{R}^2_+$ with $x \notin P(S) \cap \{0, \underline{w}(S), \bar{w}(S)\}$. ∎

The remainder of this section consists of two parts. In part A we characterize a family of finite p-solutions with the aid of the IIA-property; we call a p-solution *finite* if $\phi_S \in F(S)$ for every $S \in B$. In part B, we characterize a family of p-solutions with the aid of the so-called "conditional"-IIA-property, which is strictly weaker than IIA.

Part A : Finite probabilistic solutions with the IIA-property

We start with some additional notations. For every $k \in \mathbb{N}$, let

$Q^k := \{x \subset \mathbb{R}^k : x \geq 0, \Sigma_{i=1}^k x_i = 1\}$.

Let further $\bar{N}$ be the family of p-solutions $N \cup \{\underline{w}, \bar{w}, D\}$ (D is the *disagreement solution*, see Definition 11.7). For $k \in \mathbb{N}$, $q \in Q^k$, and $\nu = (\nu^1, \nu^2, \ldots, \nu^k) \in \bar{N}^k$, let the finite p-solution $q.\nu$ be defined by $q.\nu_S(E) := \Sigma_{i=1}^k q_i \nu_S^i(E)$ for every $S \in B$ and $E \in \sigma(S)$. In words, $q.\nu$ is a probability mixture of k p-solutions of the family $\bar{N}$. Our main result in part A is the following theorem.

Theorem 22.5. ϕ is a finite p-solution satisfying IR, STI, and IIA, if and only if $\phi = q.\nu$ for some $k \in \mathbb{N}$, $q \in Q^k$, $\nu \in \bar{N}^k$.

Note that Theorem 11.8 is a special case of Theorem 22.5. See also the paragraph before Def. 11.7, about the disagreement solution D.

In order to prove the theorem, we need some lemmas. But first we give the following proposition.

Proposition 22.6. For every $k \in \mathbb{N}$, $q \in Q^k$, $\nu \in \bar{N}^k$, the p-solution $q.\nu$ is finite and satisfies IR, STI, and IIA.

Proof. Follows from the fact that every $\nu \in \bar{N}$ satisfies the mentioned properties. ∎

For the following lemma, recall (Def. 9.12) that, for $S \in B$, the function $f_S^1 : [0, \bar{w}_2(S)] \to \mathbb{R}$ has graph $\underline{W}(S) \cup P(S) \cap \mathbb{R}_+^2$ and $f_S^2 : [0, \underline{w}_1(S)] \to \mathbb{R}$ has graph $\bar{W}(S) \cup P(S) \cap \mathbb{R}_+^2$. Both functions are nonincreasing and concave (cf. Lemma 9.13).

<u>Lemma 22.7.</u> Let $S \in B$. Then :

(i) There are at most countably many points in the domains of f_S^1 and f_S^2 where these functions are not differentiable.

(ii) For every $x \in P(S) \cap \mathbb{R}_+^2$ there is a $\phi \in N$ with $\phi(S) = x$; if $x > 0$, $x \neq \bar{p}(S)$, then this ϕ is unique if and only if f_S^1 is differentiable in x_2; if $x > 0$, $x \neq \underline{p}(S)$, then this ϕ is unique if and only if f_S^2 is differentiable in x_1.

(iii) For every $x \in P(S) \cap \mathbb{R}_+^2$, there is a closed interval J_x in $[0,1]$ such that, for all $\phi \in N$, $\phi(S) = x$ if and only if $\phi = N^t$ for some $t \in J_x$.

(iv) If $x, y \in P(S) \cap \mathbb{R}_+^2$, then $\alpha \leq \beta$ for all $\alpha \in J_x$, $\beta \in J_y$ if and only if $x_1 \leq y_1$.

<u>Proof.</u> (i) E.g. Theorem 25.3 in Rockafellar (1970). (ii) Let $x \in P(S) \cap \mathbb{R}_+^2$. If $x = \bar{p}(S)$ then $x = D^2(S)$. If $x = \underline{p}(S)$ then $x = D^1(S)$. Otherwise there is a supporting line of S at x (e.g. Theorem 11.6 in Rockafellar (1970)) with a strictly positive normal vector, say $(\frac{t}{x_1}, \frac{1-t}{x_2})$ for some $t \in (0,1)$. Then $x = N^t(S)$ by Lemma 12.1. Now let $x > 0$, $x \neq \bar{p}(S)$. If $(f_S^1)'(x_2)$ exists then : either $(f_S^1)'(x_2) = 0$ implying $\phi(S) = x \Leftrightarrow \phi = D^1$ for all $\phi \in N$ by Lemma 12.1; or $(f_S^1)'(x_2) < 0$ implying $\phi(S) = x \Leftrightarrow \phi = N^t$ for all $\phi \in N$ and some unique $t \in (0,1)$, by Lemma 12.1 again. If f_S^1 is not differentiable in x_2, then $N^t(S) = x$ for infinitely many $t \in (0,1)$, by Lemma 12.1 again. The final statement in (ii) can be proved analogously. Also (iii) and (iv) can be proved mainly with the aid of Lemma 12.1. We only note : if $x = \bar{p}(S) = \bar{w}(S)$, then $J_x = \{0\}$; if $x = \bar{p}(S) \neq \bar{w}(S)$ then $J_x = [0,t]$ for some $t \in [0,1]$; if $x \in P(S)$ with $\bar{p}_1(S) < x_1 < \underline{p}_1(S)$ then $J_x = [s,t]$ for some $s, t \in (0,1)$ with $s \leq t$; if $x = \underline{p}(S) \neq \underline{w}(S)$, then $J_x = [t,1]$ for some $t \in [0,1]$; and if $x = \underline{p}(S) = \underline{w}(S)$, then $J_x = \{1\}$. The proof of these facts is left to the reader. ■

If we have a finite p-solution ϕ satisfying IR, STI, and IIA, and we know that $\phi = q.\nu$ for some $k \in \mathbb{N}$, $q \in Q^k$, $\nu \in \bar{N}^k$, then, according to Lemma 22.7, the exact values of k, q, and ν, could be determined by considering

ϕ_S for an $S \in B$ with only positive Pareto optimal points and with Pareto functions f^1_S and f^2_S differentiable everywhere on the interiors of their domains; e.g. the comprehensive hull of the ball in $\mathbb{R}^2$ with center (1,1) and radius 1. For proof-technical reasons, however, it will be more convenient to look at the games $\Delta = \text{comv}\{(1,0),(0,1)\}$ and $\square = \text{com}\{(1,1)\}$.

Lemma 22.8. Let ϕ be a finite p-solution satisfying IR, STI, and IIA. Then : (i) $\phi_\Delta(0) = \phi_\square(0)$, (ii) $\phi_\square((1,1)) = \phi_\Delta(P(\Delta)) - \phi_\square((1,0)) - \phi_\square((0,1))$, (iii) $\phi_\Delta((1,0)) \geqq \phi_\square((1,0))$, $\phi_\Delta((0,1)) \geqq \phi_\square((0,1))$.

Proof. By IIA, $\phi_\Delta(0) \geqq \phi_\square(0)$, and by IIA and STI, $\phi_\square(0) = \phi_{\frac{1}{2}\square}(0) \geqq \phi_\Delta(0)$, hence (i) is proved. (iii) follows by IIA. In view of Lemma 22.4 and (i), $\phi_\Delta(P(\Delta)) = 1 - \phi_\Delta(0) = 1 - \phi_\square(0) = \phi_\square((1,1)) + \phi_\square((1,0)) + \phi_\square((0,1))$, hence (ii) is proved. ■

For an arbitrary fixed p-solution ϕ, we define the following numbers in [0,1] :

$q_D := \phi_\square(0)$, $q_{\underline{w}} := \phi_\square((1,0))$, $q_{\bar{w}} := \phi_\square((0,1))$, $q_1 := \phi_\Delta((1,0)) - \phi_\square((1,0))$, $q_0 := \phi_\Delta((0,1)) - \phi_\square((0,1))$, $q_t := \phi_\Delta((t,1-t))$ for every $t \in (0,1)$. If ϕ is a finite p-solution satisfying IR, STI, and IIA, then, by Lemma 22.8 and Lemma 22.4, we know :

(22.1) $\quad \phi_S(E) = q_D D_S(E) + q_{\underline{w}} \underline{w}_S(E) + q_{\bar{w}} \bar{w}_S(E) + \Sigma_{t\in[0,1]} q_t N^t_S(E)$ for each $E \in \sigma(S)$

holds for $S = \square$ and $S = \Delta$. Until further notice, ϕ will be this arbitrary but fixed finite p-solution satisfying IR, STI, and IIA. We want to show that ϕ satisfies (22.1) for every $S \in B$.

Lemma 22.9. Let $S \in B$. Then : (i) $\phi_S(N^t(S)) \geqq q_t$ for every $t \in (0,1)$, (ii) $\phi_S(0) \geqq q_D$, (iii) $\phi_S(\underline{w}(S)) \geqq q_{\underline{w}}$, (iv) $\phi_S(\bar{w}(S)) \geqq q_{\bar{w}}$, (v) $\phi_S(N^t(S)) \geqq q_1$, (vi) $\phi_S(N^0(S)) \geqq q_0$.

Proof. (ii), (iii), and (iv), follow, with the aid of Lemma 22.3 (ii) from applying STI and IIA to S and $h(S)\square$. (i) follows, with the aid of Lemma 22.3 (ii) and Lemma 12.1, from applying, for each $t \in (0,1)$, STI and IIA to S and $(t^{-1}N^t_1(S), (1-t)^{-1}N^t_2(S))\Delta$. Of (v) and (vi), we prove (vi). If $S = \square$, then (vi) holds since (22.1) holds for $S = \square$. Now suppose $S \neq \square$. Since $\phi_S \in F(S)$, there is a point $z \in P(S)$, $z \neq N^0(S)$ such that $\phi_S(x) = 0$ for all $x \in P(S)$ with $N^0_1(S) < x_1 \leqq z_1$. Let the straight line through $\bar{w}(S)$ and z

contain $(\alpha,0)$ where $\alpha > 0$, and let $T := \text{comv}\{\bar{w}(S),(\alpha,0)\}$. Let $V := S \cap T$. By STI, $\phi_T(\bar{w}(S)) = q_0 + q_{\bar{w}}$, hence by IIA, $\phi_V(\bar{w}(S)) \geq q_0 + q_{\bar{w}}$. We now distinguish two cases. First, suppose $\bar{w}(S) \neq N^0(S)$. By the choice of z and Lemma 22.4, $\phi_S(V) = 1 - \phi_S(N^0(S))$. Further, $\phi_S(V) = \phi_S(\bar{w}(S)) + \phi_S(V\setminus\{\bar{w}(S)\})$, so

$\phi_S(\bar{w}(S)) = 1 - \phi_S(N^0(S)) - \phi_S(V\setminus\{\bar{w}(S)\}) \geq 1 - \phi_S(N^0(S)) - \phi_V(V) + \phi_V(\bar{w}(S)) \geq q_{\bar{w}} + q_0 - \phi_S(N^0(S))$. Further, STI and IIA give $q_{\bar{w}} = \phi_{a\Box}(\bar{w}(S)) \geq \phi_S(\bar{w}(S))$ where $a = N^0(S)$. We conclude, for this case, that $\phi_S(N^0(S)) \geq q_0$. Secondly, if $\bar{w}(S) = N^0(S)$ then $1 - \phi_S(N^0(S)) = \phi_S(V\setminus\{\bar{w}(S)\}) \leq \phi_V(V\setminus\{\bar{w}(S)\}) \leq 1 - q_0 - q_{\bar{w}}$. So also in this case : $\phi_S(N^0(S)) \geq q_0$. ■

If $S \in B$ is such that the set $\{\bar{w}(S),\underline{w}(S)\} \cup \{N^t(S) : q_t > 0\}$ contains exactly $2 + |\{t \in [0,1] : q_t > 0\}|$ elements, then ϕ satisfies (22.1) for such an S, in view of Lemma 22.9, since the sum of all probabilities $(q_D, q_{\underline{w}}, \ldots)$ is 1. The following two lemmas take care of games S where this is not the case.

<u>Lemma 22.10.</u> Let $S \in B$.

(i) If $\bar{w}(S) = N^0(S)$, then $\phi_S(\bar{w}(S)) \geq q_{\bar{w}} + q_0$.

(ii) If $\underline{w}(S) = N^1(S)$, then $\phi_S(\underline{w}(S)) \geq q_{\underline{w}} + q_1$.

<u>Proof.</u> We only prove (i). Let $\bar{w}(S) = N^0(S)$. Choose $z > 0$ as in the proof of Lemma 22.9. Let $V := \{x \in S : x_2 \leq z_2\}$. Then, by IIA, $\phi_V(V\setminus\{z,(0,z_2)\}) \geq \phi_S(V\setminus\{z,(0,z_2)\}) = \phi_S(S\setminus\{\bar{w}(S)\}) = 1 - \phi_S(\bar{w}(S))$. So $1 = \phi_V(V) \geq 1 - \phi_S(\bar{w}(S)) + \phi_V(\{z,(0,z_2)\}) \geq 1 - \phi_S(\bar{w}(S)) + q_0 + q_{\bar{w}}$, where the last inequality follows from Lemma 22.9, (iv) and (vi). We conclude that $\phi_S(\bar{w}(S)) \geq q_{\bar{w}} + q_0$. ■

<u>Lemma 22.11.</u> Let $S \in B$, $z \in P(S)$, $z > 0$. Then : $\phi_S(z) \geq \sum_{t\in[0,1]} q_t 1_{\{N^t(S)\}}(z)$.

<u>Proof.</u> Let $I := \{N^t \in N : N^t(S) = z, q_t > 0\}$. If $|I| = 1$, then the proof is complete in view of Lemma 22.9, (i), (v), and (vi). If $N^0, N^1 \in I$, then $S = h(S)\Box$ (e.g., Lemma 22.7 (iii)), and the proof is complete again. We are left with the case : $|I| \geq 2$ and $\{N^0,N^1\} \not\subset I$, and we will give a proof by induction on $|I|$. So we suppose the lemma holds

for $|I| < k$, where $k \in \mathbb{N}$, $k \geqq 2$. Then let $|I| = k$, $I \not\ni \{N^0, N^1\}$. Let J_z be the closed interval in $[0,1]$ as in Lemma 22.7 (iii). There is an $r \in \text{int}(J_z)$ such that $t' < r < t''$, with $t', t'' \in J_z$ and $N^{t'}, N^{t''} \in I$. By Lemma 12.1, there is a supporting line ℓ of S at z with equation $rz_2x_1 + (1-r)z_1x_2 = z_1z_2$. Since $r \in \text{int}(J_z)$, $\ell \cap S = \{z\}$. Let $\varepsilon > 0$, $\varepsilon < z_1z_2$. Then :

$S^{\varepsilon} := S \cap \{x \in \mathbb{R}^2 : rz_2x_1 + (1-r)z_1x_2 \leqq z_1z_2 - \varepsilon\} \in B$.

By Lemma 12.1, $N^r(S^{\varepsilon}) = (z_1z_2 - \varepsilon)(z_1z_2)^{-1}z$. Since $z > N^r(S^{\varepsilon})$, we have $N^r(S^{\varepsilon}) \notin P(S)$, so, in particular, $f^1_{S^{\varepsilon}}$ is differentiable in $N^r_2(S^{\varepsilon})$. Hence, it follows by Lemma 22.7, (ii) - (iv), that :

(22.2) $N^t_1(S^{\varepsilon}) < [>]\ N^r_1(S^{\varepsilon})$ if $N^t \in I$ with $t < [>] r$, $N^1_1(S^{\varepsilon}) > N^r_1(S^{\varepsilon})$, $N^0_1(S^{\varepsilon}) < N^r_1(S^{\varepsilon})$.

Since $\ell \cap S = \{z\}$ and ϕ is a finite p-solution, there is an $\eta \in (0, z_1z_2)$ small enough such that

(22.3) $\phi_S(S \setminus \{z\}) = \phi_S(S^{\eta}) = \phi_S(S^{\eta} \setminus A)$

where $A := \{x \in S^{\eta} : rz_2x_1 + (1-r)z_1x_2 = z_1z_2 - \eta\}$.

By definition of S^{η}, we have $\psi(S^{\eta}) \in A$ for all $\psi \in I$. By the choice of r, (22.2), and the induction hypothesis, we obtain :

(22.4) $\phi_{S^{\eta}}(A) \geqq \alpha := \sum_{N^t \in I} q_t$.

By (22.4) we have : $1 = \phi_{S^{\eta}}(S^{\eta}) = \phi_{S^{\eta}}(S^{\eta} \setminus A) + \phi_{S^{\eta}}(A) \geqq \alpha + \phi_{S^{\eta}}(S^{\eta} \setminus A)$.

So by (22.3) : $\phi_S(S \setminus \{z\}) \leqq 1 - \alpha$, hence : $\phi_S(z) \geqq \alpha$, which proves the lemma. ■

Proof of Theorem 22.5. The "if"-part of the theorem is Proposition 22.6. For the "only if"-part, let ϕ be a finite p-solution with the properties IR, STI, and IIA. We want to show : $\phi = q.\nu$ for some $k \in \mathbb{N}$, $q \in Q^k$, $\nu \in \bar{N}^k$. Let q_D, $q_{\bar{w}}$, $q_{\underline{w}}$, q_t, be the numbers defined before Lemma 22.9. Among these numbers there are only finitely many positive ones, say k, and these sum to 1, in view of Lemma 22.8. These positive numbers can be arranged to constitute a vector $q \in Q^k$, and the corresponding solutions constitute an element $\nu \in \bar{N}^k$. Lemmas 22.9 - 22.11 show that ϕ satisfies (22.1) for every $S \in B$. Hence, $\phi = q.\nu$. ■

We conclude this part of section 22 by noting that Theorem 11.8 can be derived from Theorem 22.5 by considering p-solutions corresponding to bargaining solutions; recall that Theorem 11.8 provides a characterization of the family N of bargaining solutions with the aid of IIA.

Part B : Conditional independence of irrelevant alternatives

In this second half of the section, we concern ourselves with a variation on Theorem 22.5 : we will not require a p-solution ϕ to be finite but, on the other hand, replace IIA by a stronger property. This property bears close resemblance to the so-called Choice Axiom in Luce (1979).

Definition 22.12. We call the p-solution ϕ *conditionally independent of irrelevant alternatives* if for all $S, T \in B$ with $S \subset T$ and all $E \in \sigma(S)$, we have : $\phi_S(E)\phi_T(S) = \phi_T(E)$.
(Conditional independence of irrelevant alternatives : CIIA)

An equivalent way to formulate CIIA is : for all S and T in B with $S \subset T$, and for all $E \in \sigma(S)$, if $\phi_T(S) \neq 0$, then $\phi_S(E) = \phi_T(E)\phi_T(S)^{-1}$. So $\phi_S(E)$ is equal to the conditional probability of E given S under ϕ_T. This explains the use of the expression "conditional" IIA.

Note that, for S and T in B with $S \subset T$, and $E \in \sigma(S)$, IIA only requires $\phi_S(E) \geq \phi_T(E)$. If $\phi_T(T \setminus S) \neq 0$, then this remaining "probability mass" has to be distributed over S. The CIIA-property gives one way to do this. Thus, IIA is weaker than CIIA; in the sequel, it will turn out that IIA is strictly weaker, even in the presence of conditions like PO, IR, and STI.

The p-solutions described in the following definition all satisfy IR, STI, and CIIA.

Definition 22.13. For every $t \in (0,\infty)$, the solutions $\underline{D}^t$, $\bar{D}^t$, $\underline{W}^t$, $\bar{W}^t$, are defined as follows. For $S \in B$ and $E \in \sigma(S)$,

$$\underline{D}_S^t(E) := \underline{w}_1(S)^{-t} \int_{[0,\underline{w}_1(S)]} 1_{\{x \in E\,:\, x_2=0\}}(x)\,dx_1^t,$$

$$\bar{D}_S^t(E) := \bar{w}_2(S)^{-t} \int_{[0,\bar{w}_2(S)]} 1_{\{x \in E\,:\, x_1=0\}}(x)\,dx_2^t,$$

$$\underline{W}_S^t(E) := \begin{cases} \underline{p}_2(S)^{-t} \int\limits_{[0,\underline{p}_2(S)]} 1_{\{x \in E:\, x_1 = \underline{p}_1(S)\}}(x)\,dx_2^t & \text{if } \underline{p}_2(S) > 0 \\ \underline{w}_S(E) \text{ if } \underline{p}_2(S) = 0, \end{cases}$$

$$\bar{W}_S^t(E) := \begin{cases} \bar{p}_1(S)^{-t} \int\limits_{[0,\bar{p}_1(S)]} 1_{\{x \in E:\, x_2 = \bar{p}_2(S)\}}(x)\,dx_1^t & \text{if } \bar{p}_1(S) > 0 \\ \bar{w}_S(E) \text{ if } \bar{p}_1(S) = 0. \end{cases}$$

For all $t,s \in (0,\infty)$, the solutions $h^{t,s}$ are defined as follows. For $S \in B$, $E \in \sigma(S)$:

$h_S^{t,s}(E) := \alpha \iint_{\mathbb{R}_+^2} 1_E(x)\,dx_1^t dx_2^s$ where $\alpha := (\iint_{\mathbb{R}_+^2} 1_S(x)\,dx_1^t dx_2^s)^{-1}$.

So, for $S \in B$, $\underline{D}_S^t$ and $\bar{D}_S^t$ are nonatomic probability measures with supports $\text{conv}\{0,\underline{w}(S)\}$ and $\text{conv}\{0,\bar{w}(S)\}$ respectively, and $\underline{W}_S^t$ and $\bar{W}_S^t$ are nonatomic probability measures with supports $\underline{W}(S)$ and $\bar{W}(S)$ (if $\underline{W}(S) \neq \{\underline{w}(S)\}$, $\bar{W}(S) \neq \{\bar{w}(S)\}$), respectively, and $h_S^{t,s}$ is a nonatomic probability measure with support $S \cap \mathbb{R}_+^2$.

The proof of the following proposition is left to the reader.

<u>Proposition 22.14.</u> If $\phi \in \bar{N}$ or $\phi \in \{\underline{D}^t, \bar{D}^t, \underline{W}^t, \bar{W}^t, h^{t,s} : t,s \in (0,\infty)\}$, then ϕ satisfies IR, STI, and CIIA.

We will show that the converse of this proposition also holds. The proof will be based upon a string of lemmas. In these lemmas, ϕ is an arbitrary but fixed p-solution satisfying IR, STI, and IIA.

<u>Lemma 22.15.</u> Let $E \in \sigma(\Box)$, $E \subset \mathbb{R}_+^2$, and let $a^1, a^2, \ldots \in \mathbb{R}_{++}^2$ with $(1,1) \geq a^1 \geq a^2 \geq \ldots$. Let $a^n E = E$ for every $n \in \mathbb{N}$, and $E = (\lim_{n\to\infty} a^n)\Box \cap \mathbb{R}_+^2$. Then, if $\phi_\Box(E) > 0$, we have $\phi_\Box(E) = 1$.

<u>Proof.</u> Let $\phi_\Box(E) = \varepsilon > 0$. By STI and CIIA, we have for every $n \in \mathbb{N}$: $\varepsilon = \phi_\Box(E) = \phi_{a^n\Box}(E)\phi_\Box(a^n\Box) = \varepsilon\phi_\Box(a^n\Box)$, hence $\phi_\Box(a^n\Box) = 1$. So

$\lim_{n\to\infty} \phi_\Box(a^n\Box) = 1$, which implies, by σ-additivity of $\phi_\Box$, that $\phi_\Box(E) = 1$. ■

Lemma 22.16. (i) $\phi_\square(0) > 0 \Rightarrow \phi_\square(0) = 1$ (ii) $\phi_\square((1,0)) > 0 \Rightarrow \phi_\square((1,0)) = 1$ (iii) $\phi_\square((0,1)) > 0 \Rightarrow \phi_\square((0,1)) = 1$ (iv) $\phi_\square((1,1)) > 0 \Rightarrow \phi_\square((1,1)) = 1$.

Proof. (i) Apply Lemma 22.15 with $E = \{0\}$ and $a^n = (n^{-1},n^{-1})$ for every $n \in \mathbb{N}$. (ii) Suppose $0 < \varepsilon = \phi_\square((1,0))$. Let $0 < a < (1,1)$. By STI, CIIA, and Lemma 22.4, $0 = \phi_\square((a_1,0)) = \phi_\square(a\square)\phi_{a\square}((a_1,0)) = \varepsilon\phi_\square(a\square)$, hence $\phi_\square(a\square) = 0$. From this we may conclude $\phi_\square(\text{conv}\{(0,0),(1,0)\} \setminus \{(1,0)\}) = 0$. Now apply Lemma 22.15 with $E = \text{conv}\{(0,0),(1,0)\}$ and $a^n = (1,n^{-1})$ for every $n \in \mathbb{N}$. This gives $\phi_\square(\text{conv}\{(0,0),(1,0)\}) = 1$, hence $\phi_\square((1,0)) = 1$. (iii) Analogous to (ii). (iv) Analogously as in (ii), one proves $\phi_\square(a\square) = 0$ for every $a \in \mathbb{R}^2_{++}$ with $a \leqq (1,1)$, $a \neq (1,1)$. Hence, $\phi_\square(\square\setminus\{(1,1)\}) = \phi_\square(\bigcup_{a\in A} a\square) = 0$ where $A := \{a \in \mathbb{Q}^2_{++} : a \leqq (1,1),\ a \neq (1,1)\}$. So $\phi_\square((1,1)) = 1$. ■

For every $S \in B$, we denote $\underline{D}(S) := \text{conv}\{(0,0),\underline{w}(S)\}$, and $\bar{D}(S) := \text{conv}\{(0,0),\bar{w}(S)\}$. ($\underline{D}$, $\bar{D} : B \to \mathbb{R}^2$ are multisolutions.)

Lemma 22.17. (i) $\phi_\square(\text{relint}(\underline{D}(\square))) = \varepsilon > 0 \Rightarrow \varepsilon = 1$.
(ii) $\phi_\square(\text{relint}(\bar{D}(\square))) = \varepsilon > 0 \Rightarrow \varepsilon = 1$.

Proof. (i) Suppose $0 < \varepsilon = \phi_\square(\text{relint}(\underline{D}(\square)))$. By Lemma 22.15 with $E = \underline{D}(\square)$ and $a^n = (1,n^{-1})$ for every $n \in \mathbb{N}$, we have $\phi_\square(\underline{D}(\square)) = 1$. Since $\varepsilon > 0$ we have in view of Lemma 22.16 (i), (ii) : $\varepsilon = 1$.
(ii) Analogous to (i). ■

Lemma 22.18. (i) $\phi_\square(\text{relint}(\underline{W}(\square))) = \varepsilon > 0 \Rightarrow \varepsilon = 1$.
(ii) $\phi_\square(\text{relint}(\bar{W}(\square))) = \varepsilon > 0 \Rightarrow \varepsilon = 1$ (iii) $\phi_\square(\text{int}(\square \cap \mathbb{R}^2_+)) = \varepsilon > 0 \Rightarrow \varepsilon = 1$.

Proof. (i) Suppose $0 < \varepsilon < 1$. In view of Lemma 22.16 (ii), (iv), there must exist $\alpha,\delta > 0$ with $\alpha < 1$ such that $\phi_\square((\beta,1)\square) \geqq \delta$ for every β with $\alpha \leqq \beta < 1$. Hence, for every such β, we obtain by STI and CIIA :
$\phi_\square(\text{relint}(\underline{W}((\beta,1)\square))) \geqq \varepsilon\delta$. From this : $1 = \phi_\square(\square) \geqq \sum_{\alpha<\beta<1,\beta\in\mathbb{Q}} \varepsilon\delta = \infty$, an impossibility. So $\varepsilon = 1$.
(ii) Analogous to (i).
(iii) Follows from (i), (ii), and Lemmas 22.16, 22.17. ■

So far, we have examined $\text{supp}(\phi_\square)$. Next we consider ϕ_Δ.

Lemma 22.19. $\phi_{\square}((1,1)) = 1 \Rightarrow \phi_{\Delta}((t,1-t)) = 1$ for some $t \in [0,1]$.

Proof. Let $a \in \mathbb{R}^2_{++}$ with $a_1 + a_2 < 1$. By STI, CIIA, and Lemma 22.4, $0 = \phi_{\Delta}(a) = \phi_{a^{\square}}(a)\phi_{\Delta}(a^{\square}) = \phi_{\Delta}(a^{\square})$ since by assumption $\phi_{\square}((1,1)) = 1$. Then $1 = \phi_{\Delta}(\Delta) = \phi_{\Delta}(\Delta \setminus P(\Delta)) + \phi_{\Delta}(P(\Delta)) = \phi_{\Delta}(\bigcup_{a \in A} a^{\square}) + \phi_{\Delta}(P(\Delta)) = \phi_{\Delta}(P(\Delta))$, where $A := \{a \in Q^2_{++} : a_1 + a_2 < 1\}$. So $\phi_{\Delta}(P(\Delta)) = 1$.

Suppose $\phi_{\Delta}((t,1-t)) \neq 1$ for every $t \in [0,1]$. Then we can find $s,u \in [0,1]$ with $s < u$ such that $\phi_{\Delta}(\{x \in P(\Delta) : x_1 \leq s\}) \neq 0$, $\phi_{\Delta}(\{x \in P(\Delta) : x_1 \geqq u\}) \neq 0$. (See Fig. 22.1.) Take $r \in (s,u)$. Take $\eta \in (r,1)$ close enough to 1 to guarantee that $b_2(1-u) < 1-r$ where $(b_1,b_2) \in \mathbb{R}^2_{++}$ such that $b\Delta = \Delta^{\eta}$, where Δ^{η} is th convex comprehensive hull of $(\eta,0)$ and the point with first coordinate 0 on the straight line through $(\eta,0)$ and $(r,1-r)$. Further, let $T := \Delta \cap \Delta^{\eta}$, so $T = \text{comv}\{(0,1),(r,1-r),(\eta,0)\}$. Since $\phi_{\Delta}(P(\Delta)) = 1$, we have $\phi_{\Delta}(T) = \phi_{\Delta}(\text{conv}\{(0,1),(r,1-r)\})$. By CIIA : $\phi_{T}(\text{conv}\{(0,1),(r,1-r)\})\phi_{\Delta}(T) = \phi_{\Delta}(\text{conv}\{(0,1),(r,1-r)\}) \geqq \phi_{\Delta}(\text{conv}\{(0,1),(s,1-s)\}) \neq 0$, hence $\phi_{T}(\text{conv}\{(0,1),(r,1-r)\}) = 1$. From this, we obtain : $\phi_{T}(b\{x \in P(\Delta): x_1 \geqq u\}) = 0$, in view of the choice of η. On the other hand, by CIIA and STI, we have $\phi_{T}(b\{x \in P(\Delta) : x_1 \geq u\})\phi_{\Delta^{\eta}}(T) = \phi_{\Delta^{\eta}}(b\{x \in P(\Delta) : x_1 \geqq u\}) = \phi_{\Delta}(\{x \in P(\Delta) : x_1 \geqq u\}) \neq 0$, hence $\phi_{T}(b\{x \in P(\Delta) : x_1 \geqq u\}) \neq 0$. We have a contradiction and may conclude that $\phi_{\Delta}((t,1-t)) = 1$ for some $t \in [0,1]$. ■

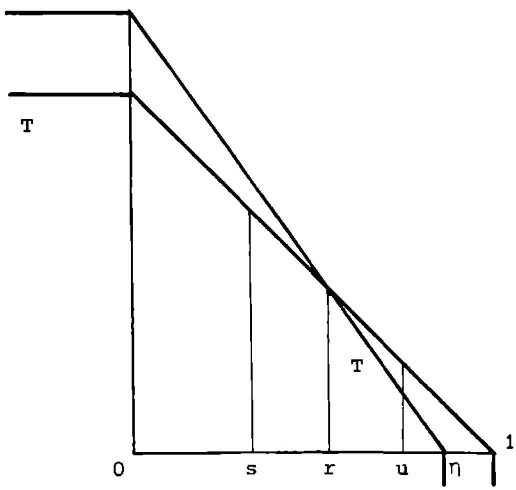


Figure 22.1.

Before proceeding, we need an elementary result in real analysis. The reader may prove it for himself, or, otherwise, the result may be derived from, e.g., Theorem 2.6.3 in Eichhorn (1978).

Lemma 22.20. For the function $f : [0,1] \to [0,1]$, the following two statements are equivalent :

(i) There exists $t \in (0,\infty)$ with $f(x) = x^t$ for all $x \in [0,1]$.

(ii) The function f has the following properties : (a) $f(x) = 0$ if and only if $x = 0$, for all $x \in [0,1]$ (b) f is continuous in 0 and 1, and f is bounded (c) $f(xy) = f(x)f(y)$ for all $x,y \in [0,1]$.

We take up again the string of lemmas leading to the converse of Proposition 22.14. Again, ϕ is a solution satisfying IR, STI, and CIIA.

Lemma 22.21. (i) If $\phi_\square(\text{relint}(\underline{D}(\square))) = 1$, then there is a $t > 0$ such that $\phi_\square(\text{conv}\{(0,0),(\lambda,0)\}) = \lambda^t$ for all $\lambda \in [0,1]$. (ii) If $\phi_\square(\text{relint}(\bar{D}(\square))) = 1$, then there is a $t > 0$ such that $\phi_\square(\text{conv}\{(0,0),(0,\lambda)\}) = \lambda^t$ for all $\lambda \in [0,1]$. (iii) If $\phi_\square(\text{relint}(\underline{W}(\square))) = 1$, then there is a $t > 0$ such that $\phi_\square(\text{conv}\{(1,0),(1,\lambda)\}) = \lambda^t$ for all $\lambda \in [0,1]$. (iv) If $\phi_\square(\text{relint}(\bar{W}(\square))) = 1$, then there is a $t > 0$ such that $\phi_\square(\text{conv}\{(0,1),(\lambda,1)\}) = \lambda^t$ for all $\lambda \in [0,1]$.

Proof. We only prove (i). The proofs of (ii), (iii), and (iv), are analogous. Suppose $\phi_\square(\text{relint}(\underline{D}(\square))) = 1$. Let $f : [0,1] \to [0,1]$ be defined by $f(\lambda) = \phi_\square(\text{conv}\{(0,0),(\lambda,0)\})$ for all $\lambda \in [0,1]$. The proof is complete if we show that f satisfies (ii) of Lemma 22.20. Obviously, $f(0) = 0$ and $f(1) = 1 > 0$. Hence, (a) is partly proved.

Next, let $\eta,\lambda \in [0,1]$. If $\eta \in \{0,1\}$ or $\lambda \in \{0,1\}$, then $f(\eta\lambda) = f(\eta)f(\lambda)$. Suppose $0 < \lambda \leqq \eta < 1$. Then, by CIIA and STI,

$f(\lambda\eta) = \phi_\square((\eta,1)\square)\phi_{(\eta,1)\square}(\text{conv}\{(0,0),(\lambda\eta,0)\}) = f(\eta)f(\lambda)$. So (c) holds.

Now let $\hat{\lambda} := \inf\{\lambda \in [0,1] : f(\lambda) > 0\}$. Then obviously $\hat{\lambda} < 1$.

Suppose that $\hat{\lambda} > 0$. Take $\eta \in (\hat{\lambda},1)$ such that $\eta^2 < \hat{\lambda}$. Then $f(\eta^2) = 0$ and $f(\eta) > 0$, in contradiction with $f(\eta^2) = f(\eta)^2$. So $\hat{\lambda} = 0$, and (a) is proved completely.

Since f is nondecreasing and $\phi_\square((1,1)) = 0$, it follows that f is continuous in 1. Further, if λ decreases to 0, then $f(\lambda)$ decreases to $\phi_\square(0) = 0 = f(0)$, so f is continuous in 0. Noting that f is bounded, we have proved (b). ■

Lemma 22.22. Suppose $\phi_\Box(\mathrm{int}(\Box \cap \mathbf{R}^2_+)) = 1$. Then there exist $t, s \in (0,\infty)$ such that $\phi_\Box(E) = \iint_{\mathbf{R}^2_+} 1_E(\zeta,\eta)\,d\zeta^t d\eta^s$ for every $E \in \sigma(\Box)$.

Proof. Let the functions $f,g : [0,1] \to [0,1]$ be defined by $f(\zeta) = \phi_\Box((\zeta,1)\Box)$ and $g(\eta) = \phi_\Box((1,\eta)\Box)$ for all $\zeta,\eta \in [0,1]$. Analogously as in the proof of Lemma 22.21, one verifies that f and g satisfy (ii) of Lemma 22.20, so there exist $s,t \in (0,\infty)$ such that $f(\zeta) = \zeta^t$ and $g(\eta) = \eta^s$ for all $\zeta,\eta \in [0,1]$. Now let the probability measures μ_1, μ_2 on the σ-algebra $\sigma([0,1])$ be defined by $\mu_1(C) = \int_C d\zeta^t$ and $\mu_2(C) = \int_C d\eta^s$ for all $C \in \sigma([0,1])$.

Let $\hat{\zeta}$, $\hat{\eta} \in (0,1]$. Then, by CIIA and STI, we have

$$\phi_\Box((\hat{\zeta},\hat{\eta})\Box) = \phi_\Box((1,\hat{\eta})\Box)\phi_{(1,\hat{\eta})\Box}((\hat{\zeta},\hat{\eta})\Box) = g(\hat{\eta})f(\hat{\zeta}) =$$

$$= \int_{[0,\hat{\eta}]} d\eta^s \int_{[0,\hat{\zeta}]} d\zeta^t = \mu_1([0,\hat{\zeta}])\mu_2([0,\hat{\eta}]).$$

Hence, because $\sigma(\Box \cap \mathbf{R}^2_+) = \sigma([0,1]) \times \sigma([0,1])$, $\phi_\Box = \mu_1 \times \mu_2$ is the product measure of μ_1 and μ_2. So, by Fubini's theorem, we have for every $E \in \sigma(\Box)$:
$\phi_\Box(E) = \iint_{\mathbf{R}^2_+} 1_E(\zeta,\eta)\,d\zeta^t d\eta^s$. ■

We are now sufficiently equipped to show :

Proposition 22.23. $\phi \in \bar{N} \cup \{\bar{D}^t, \underline{D}^t, \bar{W}^t, \underline{W}^t, h^{t,s} : t,s \in (0,\infty)\}$.

Proof. We distinguish several cases.

(i) $\phi_\Box(0) \neq 0$. Then $\phi_\Box(0) = 1$ by Lemma 22.16 (i). Let $S \in B$. By IR and CIIA, $\phi_S(E) = \phi_{S_+}(E)$ for every $E \in \sigma(S_+)$. By STI and CIIA, $1 = \phi_\Box(0) = \phi_{h(S)\Box}(0) = \phi_{h(S)\Box}(S_+)\phi_{S_+}(0) = \phi_S(0)$. So $\phi = D$.

(ii) $\phi_\Box((1,0)) \neq 0$. Analogously as in (i), $\phi = \underline{w}$, with the aid of Lemma 22.16 (ii).

(iii) $\phi_\Box((0,1)) \neq 0$. Analogously as in (i), $\phi = \bar{w}$, with the aid of Lemma 22.16 (iii).

(iv) $\phi_\Box(\mathrm{relint}(\underline{D}(\Box))) \neq 0$. Analogously as in (i), $\phi = \underline{D}^t$ for some $t > 0$, with the aid of Lemma 22.17 (i) and Lemma 22.21 (i).

(v) $\phi_\Box(\mathrm{relint}(\bar{D}(\Box))) \neq 0$. Analogously as in (i), $\phi = \bar{D}^t$ for some $t > 0$, with the aid of Lemma 22.17 (ii) and Lemma 22.21 (ii).

(vi) $\phi_\Box(\mathrm{relint}(\underline{W}(\Box))) \neq 0$. Then $\phi_\Box(\mathrm{relint}(\underline{W}(\Box))) = 1$ by Lemma 22.18(i). Let $S \in B$. We distinguish two subcases.

(vi.a) $\underline{p}(S) \neq \underline{w}(S)$. Analogously as in (i), $\phi_S = \underline{W}_S^t$, for $t > 0$ with $\phi_\square = \underline{W}_\square^t$, with the aid of Lemma 22.21 (iii).

(vi.b) $\underline{p}(S) = \underline{w}(S)$. Suppose $\phi_S(\underline{p}(S)) \neq 1$. Then there exists $z \in P(S)$, $z > 0$, such that for all $y \in P(S)$ with $z_1 \leq y_1 < \underline{p}_1(S)$, we have $\phi_S(S_y) \geq \phi_S(S_z) > 0$ where $S_y := \{x \in S : x_1 \leq y_1\}$. By (vi.a), $\phi_S(S_y) = \phi_S(\underline{W}(S_y))$, and since $y \neq y' \Rightarrow \underline{W}(S_y) \cap \underline{W}(S_{y'}) = \emptyset$ for all y,y' with $z_1 \leq y_1, y_1' < \underline{p}_1(S)$, we obtain $\phi_S(S) \geq \sum_{y \in A} \phi_S(S_y) = \infty$, where $A := \{y \in \mathbb{R}^2_{++} : y \in P(S), z_1 \leq y_1 < \underline{p}_1(S)\}$. From this impossibility we conclude $\phi_S(\underline{p}(S)) = 1$.

By (vi.a) and (vi.b) we conclude : $\phi = \underline{W}^t$, for t as in (vi.a).

(vii) $\phi_\square(\mathrm{relint}(\bar{W}(\square))) \neq 0$. Analogously as in (vi), $\phi = \bar{W}^t$ for some $t > 0$, with the aid of Lemma 22.18 (ii) and Lemma 22.21 (iv).

(viii) $\phi_\square(\mathrm{int}(\square \cap \mathbb{R}^2_+)) \neq 0$. Analogously as in (i), $\phi = h^{t,s}$ for some $t,s > 0$, with the aid of Lemma 22.18 (iii) and Lemma 22.22.

(ix) and final case : $\phi_\square((1,1)) \neq 0$. Then $\phi_\square((1,1)) = 1$ by Lemma 22.16 (iv), so $\phi_\Delta((t,1-t)) = 1$ for some $t \in [0,1]$ by Lemma 22.19. If $0 < t < 1$, then $\phi = N^t$, by a modification of the proof of Theorem 11.5. If $t = 0$ [$t = 1$], then $\phi = D^2$ [$\phi = D^1$], by a modification of the proof of Theorem 11.8. ∎

Combining Propositions 22.14 and 22.23, we obtain the main result of part B of this section.

<u>Theorem 22.24.</u> A probabilistic solution ϕ satisfies IR, STI, and CIIA, if and only if $\phi \in \bar{N} \cup \{\underline{D}^t, \bar{D}^t, \underline{W}^t, \bar{W}^t, h^{t,s} : t,s \in (0,\infty)\}$.

We conclude with a few remarks.

<u>Remark 22.25.</u> The results in this section were published (without detailed proofs) in Peters and Tijs (1983). There, also a probabilistic version of the individual monotonicity property is suggested.

<u>Remark 22.26.</u> We formulate the following two properties, for a probabilistic solution ϕ.

<u>CIIA(a)</u> For all $S, T \in B$ with $S \subset T$ and every $E \in \sigma(S)$, we have :

$$\phi_S(E)\phi_T(S) \geq \phi_T(E).$$

CIIA(b) For all $S, T \in B$ with $S \subset T$ and every $E \in \sigma(S)$, we have :

$$\phi_S(E)\phi_T(S) \leqq \phi_T(E).$$

We note that both properties are equivalent to CIIA. For instance, CIIA implies CIIA(b). Let $S,T \in B$ with $S \subset T$ and $E \in \sigma(S)$, and suppose $\phi_S(E)\phi_T(S) < \phi_T(E)$. Then $\phi_T(S) > 0$ and $\phi_S(S \setminus E) = 1 - \phi_S(E) > 1 - \phi_T(E)\phi_T(S)^{-1} = \phi_T(S \setminus E)\phi_T(S)^{-1}$, hence $\phi_S(S \setminus E)\phi_T(S) > \phi_T(S \setminus E)$. This shows that CIIA(b) implies CIIA. Similarly one shows that CIIA(a) is equivalent to CIIA.

CHAPTER 8

RISK PROPERTIES

In section 5, we have introduced the relation "more risk averse than" on the family U(A) of functions defined on a non empty set of alternatives A; further, we have given a mathematical characterization of that relation (Theorem 5.5). In this chapter, we use that characterization when we study the following questions : Given a bargaining solution, is it favourable for one player in a (2-person) bargaining situation or game if the other player is replaced by a more risk averse player ? And : Which are the consequences for the latter two players ? We shall give an answer to this question by considering risk properties of bargaining solutions.

Section 23 introduces such properties, and establishes some elementary relations between them. Sections 24 and 25 give relations between risk properties and other properties, for the case where all Pareto optimal outcomes in a bargaining game correspond to riskless alternatives in the bargaining situation assumed to underly that bargaining game. The more general case, where Pareto optimal outcomes may also correspond to risky alternatives, is considered in section 26. Section 27 concludes the chapter with considerations of a strategical nature, such as : can it sometimes be advantageous to pretend to be more (or less) risk averse, for a player in a bargaining game ?

Everywhere in this chapter, attention is confined to the 2-person case : the n-person case will be considered in section 30. Most of the results in this chapter were published before, in several papers. For a survey, see Peters and Tijs (1985).

23. RISK PROPERTIES OF BARGAINING SOLUTIONS

Let $\Gamma = \langle A,\bar{a},u^1,u^2\rangle \in BS$ be a 2-person bargaining situation (see section 8). For $i = 1,2$, we denote by $C^i(\Gamma)$ the family of all nondecreasing continuous concave functions $k : u^i(A) \to \mathbb{R}$ with $k(0) = 0$ and with $k(x) > 0$ for some $x \in u^i(A)$. Further, for $i = 1,2$ and $k^i \in C^i(\Gamma)$, we denote by $K^i(\Gamma)$ the bargaining situation which arises from Γ by replacing u^i by $k^i \circ u^i$. Of course,

$K^1(\Gamma) \in BS$. In view of Theorem 5.5, we say that $K^1(\Gamma)$ arises from Γ by replacing player i by a more risk averse player.

What will be the effect on the outcome assigned by a bargaining solution if a player in a bargaining situation is replaced by a more risk averse player ? Some possible effects are described by the following properties.

<u>Definition 23.1.</u> Let $C \subset B$ and $\tilde{C} \subset CS$ (cf. notation 10.2). A 2-person bargaining solution $\phi : C \to \mathbb{R}^2$ is called *risk sensitive on* $\tilde{C}$ if, for all $i, j \in \{1,2\}$ with $i \neq j$, $\Gamma \in \tilde{C}$, $k^i \in C^i(\Gamma)$ with $K^i(\Gamma) \in CS$, we have $\phi_j(S_{K^i(\Gamma)}) \geq \phi_j(S_\Gamma)$. If $\tilde{C} = BS$ (and $C = B$) then ϕ is called *risk sensitive.*
(Risk sensitivity : RS)

<u>Definition 23.2.</u> Let C and $\tilde{C}$ be as in Def. 23.1. A 2-person bargaining solution $\phi : C \to \mathbb{R}^2$ has the *worse alternative property on* $\tilde{C}$, if, for each $i \in \{1,2\}$, $\Gamma = \langle A,\bar{a},u^1,u^2\rangle \in \tilde{C}$, $k^i \in C^i(\Gamma)$ with $K^i(\Gamma)$ in CS, $\ell \in \mathrm{alt}(\phi,\Gamma)$, $m \in \mathrm{alt}(\phi,K^i(\Gamma))$, we have $Eu^i(\ell) \geq Eu^i(m)$. If $\tilde{C} = BS$ (and $C = B$), ϕ is said to have the *worse alternative property.*
(Worse alternative : WA)

The risk sensitivity property says that it is not disadvantageous to bargain against a more risk averse opponent. It was introduced by Kihlstrom, Roth, and Schmeidler (1981), in a weaker form (since we allow non-decreasingness of k^i). The worse alternative property claims that a bargainer does not prefer an alternative giving rise to the solution outcome of the game played by his more risk averse substitute to an alternative giving rise to the solution outcome of the game played by himself. In order to illustrate the definitions so far, we give a few examples.

<u>Example 23.3.</u> Let $\Gamma \in BS$ be as in Example 8.3, and let $k^2 \in C^2(\Gamma)$, $k^2(\lambda) = \sqrt{\lambda}$ for all $\lambda \in u^2(A)$. In this case, $S_\Gamma = S_{K^2(\Gamma)} = \mathrm{comv}\{(0,0),(1,0),(0,1)\}$, so $\phi_1(S_\Gamma) = \phi_1(S_{K^2(\Gamma)})$ and $\mathrm{alt}(\phi,\Gamma) = \mathrm{alt}(\phi,K^2(\Gamma))$, $Eu^1(\ell) = Eu^1(m)$ for all $\ell,m \in \mathrm{alt}(\phi,\Gamma)$; all this holds for any bargaining solution $\phi : B \to \mathbb{R}^2$.

<u>Example 23.4.</u> Let $\Gamma \in BS$ be as in Example 8.4, and k^2 as in Example 23.3. Then $S_\Gamma = \mathrm{comv}\{(1-\lambda,\lambda) : 0 \leq \lambda \leq 1\}$, and $S_{K^2(\Gamma)} = \mathrm{comv}\{(1-\lambda,\sqrt{\lambda}) : 0 \leq \lambda \leq 1\}$.

Let $\phi : B \to \mathbb{R}^2$ be the Kalai - Smorodinsky solution KS, then : $\phi(S_\Gamma) = (\frac{1}{2},\frac{1}{2})$, $\phi(S_{K^2(\Gamma)}) = (\frac{1}{2}\sqrt{5} - \frac{1}{2},\frac{1}{2}\sqrt{5} - \frac{1}{2})$, $\mathrm{alt}(\phi,K^2(\Gamma)) = \{(\frac{1}{2}\sqrt{5} - \frac{1}{2},\frac{3}{2} - \frac{1}{2}\sqrt{5})\}$.

So $\phi_1(S_\Gamma) < \phi_1(S_{K^2(\Gamma)})$, $\frac{1}{2} = Eu^2(\ell) > \frac{3}{2} - \frac{1}{2}\sqrt{5} = u^2((\frac{1}{2}\sqrt{5} - \frac{1}{2},\frac{3}{2} - \frac{1}{2}\sqrt{5}))$ for all $\ell \in \mathrm{alt}(\phi,\Gamma)$. Also note that $[\frac{1}{2};(1,0),\frac{1}{2};(0,1)] \in \mathrm{alt}(\phi,\Gamma)$, and

$E(k^2 \circ u^2)([\frac{1}{2};(1,0),\frac{1}{2};(0,1)]) = \frac{1}{2} < \frac{1}{2}\sqrt{5} - \frac{1}{2} = k^2 \circ u^2((\frac{1}{2}\sqrt{5} - \frac{1}{2},\frac{3}{2} - \frac{1}{2}\sqrt{5}))$;
that is, the more risk averse player 2, in $K^2(\Gamma)$, prefers the alternative obtained by him (the unique element of $\mathrm{alt}(\phi,K^2(\Gamma))$, to at least one element of $\mathrm{alt}(\phi,\Gamma)$ which is the set of alternatives giving rise to the solution outcome obtained by the less risk averse player 2 in Γ.

These examples indicate, firstly, that it makes all the difference which bargaining situation is thought to underly a specific bargaining game, when we study risk properties of bargaining solutions; secondly, that a property similar to the WA property, formulated for the more risk averse player, would not always be satisfied. As to the latter remark : it may, from the point of view of the more risk averse player, actually be advantageous to be - openly - more risk averse, depending on which alternative will be picked out to realize the solution outcome. Section 27 deals with some related questions of a strategical nature.

We shall frequently use the following notation (cf. the first remark in the previous paragraph) :

$$\mathrm{BSC} := \{\Gamma = \langle A,\bar{a},u^1,u^2\rangle \in \mathrm{BS} : \text{for every } x \in P(S_\Gamma), \text{ there exists an an } a \in A \text{ with } x = (u^1(a),u^2(a))\}.$$

So BSC(C of "certain") consists of all bargaining situations in BS for which every Pareto optimal outcome is generated by a riskless alternative. Note that $\Gamma \in \mathrm{BSC}$ [$\Gamma \notin \mathrm{BSC}$] in Example 23.4 [23.3]. Bargaining situations Γ in BSC "behave nicely" under "transformations" k^i in $C^i(\Gamma)$:

Lemma 23.5. Let $\Gamma = \langle A,\bar{a},u^1,u^2\rangle \in \mathrm{BSC}$, $i \in \{1,2\}$, $k^i \in C^i(\Gamma)$. For $x \in P(S_\Gamma)$ let $\hat{x} \in \mathbb{R}^2$ denote the point with i-th coordinate $k^i(x_i)$ and j-th coordinate x_j ($j \neq i$), and let $\hat{P} := \{\hat{x} : x \in P(S_\Gamma)\}$. Then $S_{K^i(\Gamma)} = \mathrm{com}(\hat{P})$ and $P(S_{K^i(\Gamma)}) \subset \hat{P}$.

Proof. The equality follows from the inclusion. For the inclusion, let

$x \in P(S_{K^1(\Gamma)})$, say $i = 1$ and $x = (E(k^1 \circ u^1)(\ell), Eu^2(\ell))$ where $\ell \in L(A)$. Since $\Gamma \in BSC$, there exists an $a \in A$ with $(Eu^1(\ell), Eu^2(\ell)) \leq (u^1(a), u^2(a)) \in P(S_\Gamma)$. By Lemma 5.4, we obtain $(k^1 \circ u^1(a), u^2(a)) \geq (E(k \circ u^1)(\ell), Eu^2(\ell)) = x$, hence $x = (k^1 \circ u^1(a), u^2(a))$ since $x \in P(S_{K^1(\Gamma)})$. So $x \in \hat{P}$. ■

Most results on risk properties of bargaining solutions will be derived for the class BSC of bargaining situations with riskless Pareto optimal outcomes in the corresponding bargaining games. Risk properties of 2-person bargaining solutions on a larger subclass of BS will be considered in section 26. Here, we give already the following "impossibility result".

Theorem 23.6. Let $\phi : B \to \mathbb{R}^2$ be a weakly Pareto optimal and individually rational 2-person bargaining solution. Then ϕ is not risk sensitive, and does not have the worse alternative property.

Proof. Let $\Gamma = \langle A, \bar{a}, u^1, u^2 \rangle \in BS$ with $A = \{\bar{a}, a^1, a^2\}$, $u^1(\bar{a}) = u^2(\bar{a}) = 0$, $u^1(a^1) = u^2(a^2) = 2$, $u^1(a^2) = u^2(a^1) = -1$. Then $S_\Gamma = \text{comv}\{(2,-1),(-1,2)\}$ and $\phi(S_\Gamma) \in W(S_\Gamma) \cap \mathbb{R}^2_+ = \text{conv}\{(1,0),(0,1)\}$. Let, for every $\beta \in (\frac{1}{2}, 2]$, the function $k_\beta \in C^2(\Gamma)$ be defined by ·

$$k_\beta(-1) = -1,\ k_\beta(0) = 0,\ k_\beta(2) = \beta.$$

Suppose for a moment that ϕ is risk sensitive. Then $\phi_1(S_{K_\beta(\Gamma)}) \geq \phi_1(S_\Gamma)$ for all $\beta \in (\frac{1}{2}, 2]$. Since the set $W(S_{K_\beta(\Gamma)}) \cap \mathbb{R}^2_+$ shrinks to $\{0\}$ if β decreases to $\frac{1}{2}$, this implies that $\phi(S_\Gamma) = (0,1)$. By reversing the roles of the players, we similarly find that $\phi(S_\Gamma) = (1,0)$. So ϕ cannot be risk sensitive. The second statement in the theorem follows from the same example. ■

The next lemma says that, for a Pareto optimal solution, risk sensitivity implies the worse alternative property.

Lemma 23.7. Let $\phi : B \to \mathbb{R}^2$ be a Pareto optimal 2-person bargaining solution, and $\tilde{C} \subset BS$. Then, if ϕ satisfies RS on $\tilde{C}$, ϕ also satisfies WA on $\tilde{C}$.

Proof. Let $\Gamma = \langle A, \bar{a}, u^1, u^2 \rangle \in \tilde{C}$ and $k^2 \in C^2(\Gamma)$. Suppose ϕ satisfies RS on $\tilde{C}$. Then $Eu^1(\ell) \leq Eu^1(m)$ for all $\ell \in \text{alt}(\phi, \Gamma)$, $m \in \text{alt}(\phi, K^2(\Gamma))$. So by PO of ϕ, $Eu^2(\ell) \geq Eu^2(m)$ for such ℓ and m. ■

A sufficient condition for a Pareto optimal 2-person bargaining solution to be risk sensitive, is given by the following lemma.

<u>Lemma 23.8.</u> Let $\phi : B \to \mathbb{R}^2$ be a Pareto optimal bargaining solution, let $\tilde{C} \subset BS$, and suppose the following condition holds :

(23.1) For every $\Gamma = \langle A, \bar{a}, u^1, u^2 \rangle \in \tilde{C}$, $i \in \{1,2\}$, $k^i \in C^i(\Gamma)$, $\ell \in alt(\phi,\Gamma)$, $m \in alt(\phi, K^i(\Gamma))$, we have : $E(k^i \circ u^i)(\ell) \geq E(k^i \circ u^i)(m)$.

Then ϕ is risk sensitive on $\tilde{C}$.

<u>Proof.</u> With Γ, i, k^i, ℓ, m, as in (23.1), we must have $Eu^j(\ell) \leq Eu^j(m)$ for $j \neq i$, since otherwise PO of ϕ would be violated. ■

We now give three examples. The first two of these show that the converses of Lemmas 23.7 and 23.8 do not hold, at least not for general $\tilde{C}$. The third example indicates why we have not formulated condition (23.1) as a separate property.

<u>Example 23.9.</u> Let $\tilde{C}$ consist of one bargaining situation, namely $\Gamma = \langle A, \bar{a}, u^1, u^2 \rangle \in BS$ with $A = \{\bar{a}, a^1, a^2, a^3\}$, $u^1(\bar{a}) = u^1(a^2) = 0$, $u^2(\bar{a}) = u^2(a^1) = 0$, $u^1(a^1) = u^2(a^2) = 1$, $u^1(a^3) = u^2(a^3) = \frac{1}{4}$. Let $\phi : B \to \mathbb{R}^2$ be any Pareto optimal bargaining solution such that, firstly, $\phi(S_\Gamma) = (\frac{1}{2}, \frac{1}{2})$, and, secondly, for any $i \in \{1,2\}$ and $k^i \in C^i(\Gamma)$, we have $\phi_i(S_{K^i(\Gamma)}) = k^i \circ u^i(a^3)$ and $\phi_j(S_{K^i(\Gamma)}) = u^j(a^3)$ $(j \neq i)$ if thereby $\{a^3\} = alt(\phi, K^i(\Gamma))$, and $\phi(S_{K^i(\Gamma)}) = KS(S_{K^i(\Gamma)})$ otherwise (; KS is the Kalai - Smorodinsky solution). On $\tilde{C}$, such a ϕ satisfies WA but not RS.

<u>Example 23.10.</u> Let $\tilde{C}$ consist again of one bargaining situation $\Gamma = \langle A, \bar{a}, u^1, u^2 \rangle$, with $A = \{\bar{a}, a^1, a^2, a^3\}$, $u^1(\bar{a}) = u^1(a^2) = 0$, $u^2(\bar{a}) = u^2(a^1) = 0$, $u^1(a^1) = u^2(a^2) = 1$, $u^1(a^3) = u^2(a^3) = \frac{1}{2}$. Let $\phi : B \to \mathbb{R}^2$ be again any Pareto optimal bargaining solution such that $a^3 \in alt(\phi, K^i(\Gamma))$ for every $i \in \{1,2\}$ and $k^i \in C^i(\Gamma)$. Then, for every $i \in \{1,2\}$, $j \neq i$, $k^i \in C^i(\Gamma)$, we have $\phi_j(S_{K^i(\Gamma)}) = u^j(a^3) = \frac{1}{2} = \phi_j(S_\Gamma)$, so ϕ is risk sensitive on $\tilde{C}$. On the other hand, $[\frac{1}{2}; a^1, \frac{1}{2}; a^2] \in alt(\phi,\Gamma)$ and, for strictly concave k^i, $E(k^i \circ u^i)([\frac{1}{2}; a^1, \frac{1}{2}; a^2]) = \frac{1}{2} k^i \circ u^i(a^1) + \frac{1}{2} k^i \circ u^i(a^2) < k^i(\frac{1}{2}) = k^i \circ u^i(a^3) = E(k^i \circ u^i)(a^3)$, which violates (23.1). See also Example 23.4.

Example 23.11. Suppose that $\phi : B \to \mathbb{R}^2$ is a Pareto optimal bargaining solution which satisfies (23.1) on some class $\tilde{C}$ containing $\Gamma = \langle A,\bar{a},u^1,u^2 \rangle$ as in Example 23.4. Let $\phi(S_\Gamma) = (1-\lambda,\lambda)$, for some $\lambda \in [0,1]$. Then $[1-\lambda,(1,0),\lambda;(0,1)] \in \text{alt}(\phi,I)$. Let $T := \text{comv}\{(1,0),(0,1),(\frac{3}{4},\frac{3}{4})\}$. There is a $k^1 \in C^1(\Gamma)$ such that $T = S_{K^1(\Gamma)}$, so by (23.1) :

$\phi_1(T) \leq E(k^1 \circ u^1)([1-\lambda;(1,0),\lambda,(0,1)]) = (1-\lambda)k^1(1) = 1-\lambda$.

There is also a $k^2 \in C^2(\Gamma)$ with $T = S_{K^2(\Gamma)}$, which, by (23.1) again, gives : $\phi_2(T) \leq \lambda$. So $\phi(T) \leqq (1-\lambda,\lambda)$, whence either $\lambda=0$, $\phi(S_I) = (1,0)$, or $\lambda=1$, $\phi(S_I) = (0,1)$.

Example 23.11 indicates that condition (23.1) is overly strong. Because of this, we restrict attention to RS and WA.

Theorem 23.12. Let $\phi : B \to \mathbb{R}^2$ be a Pareto optimal bargaining solution. The following three statements are equivalent.

(i) ϕ satisfies RS on BSC.

(ii) ϕ satisfies WA on BSC.

(iii) For all $\Gamma = \langle A,\bar{a},u^1,u^2 \rangle \in$ BSC, $i \in \{1,2\}$, $k^i \in C^i(\Gamma)$, $a,b \in A$ with $a \in \text{alt}(\phi,\Gamma)$ and $b \in \text{alt}(\phi,K^i(\Gamma))$, we have $k^i \circ u^i(a) \geq k^i \circ u^i(b)$.

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 23.7. (ii) $\Rightarrow$ (iii) follows from the nondecreasingness of k^i. (iii) $\Rightarrow$ (i) follows from the Pareto optimality of ϕ and the definition of BSC. ■

In the next two sections, where we restrict attention to bargaining situations in BSC, we will, in view of Theorem 23.12, investigate only risk sensitivity of (Pareto optimal) solutions. The next and final theorem of this section tells us that, for Pareto optimal bargaining solutions, scale transformation invariance is a necessary condition for risk sensitivity. This result was obtained by Kihlstrom, Roth, and Schmeidler (1981).

Theorem 23.13. Let $\phi : B \to \mathbb{R}^2$ be a Pareto optimal bargaining solution satisfying RS on BSC. Then ϕ is scale transformation invariant.

Proof. Let $S \in B$, $a \in \mathbb{R}^2_{++}$. Let Γ be the trivial bargaining situation corresponding to S (see Example 9.6). Note that $\Gamma \in$ BSC, and also note that multiplying first coordinates by a_1 and by a_1^{-1} are elements of $C^1(\Gamma)$, so by RS applied twice we have : $\phi_2((a_1,1)S) = \phi_2(S)$. From this it follows by PO that

$\phi_1((a_1,1)S) = a_1\phi_1(S)$. Applying similar arguments again gives $\phi_1(aS) = \phi_1((a_1,1)S)$, $\phi_2(aS) = a_2\phi_2((a_1,1)S)$. So $\phi(aS) = a\phi(S)$. We have shown that ϕ satisfies STI. ■

24. RISK SENSITIVITY, INDEPENDENCE OF IRRELEVANT ALTERNATIVES, (GLOBAL) INDIVIDUAL MONOTONICITY

The purpose of this section is to show that the first property in its title is implied by every other property mentioned in its title (together with a few basic properties), for a bargaining solution on the class BSC (to which we restrict ourselves here, cf. Lemma 23.5, Theorem 23.6). We repeat (cf. Def. 10.3) that for a bargaining solution ϕ and a bargaining situation Γ, we sometimes write $\phi(\Gamma)$ instead of $\tilde{\phi}(\Gamma) = \phi(S_\Gamma)$. Further, in view of Lemma 5.6, we may restrict attention to cases where a $k^i \in C^i(\Gamma)$ is defined on $\text{conv}(u^i(A))$ and not only on $u^i(A)$, for $\Gamma = \langle A,\bar{a},u^1,u^2\rangle \in$ BSC and $i \in \{1,2\}$.

Theorem 24.1. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying IR, PO, STI, and IIA. Then ϕ satisfies RS on BSC.

Proof. Suppose ϕ is not risk sensitive on BSC, and let (say) $\Gamma \in$ BSC and $k^2 \in C^2(\Gamma)$ such that

(24.1) $\phi_1(K^2(\Gamma)) < \phi_1(\Gamma)$.

Let (q_1,q_2) be the point in $P(S_\Gamma)$ with $q_1 = \phi_1(K^2(\Gamma))$; such a point exists in view of Lemma 23.5. Then $q_2 > \phi_2(\Gamma) \geq 0$, in view of IR, the fact that $q \in P(S_\Gamma)$, and (24.1). In view of STI we may suppose that

(24.2) $k^2(q_2) = q_2$.

Then $\phi(K^2(\Gamma)) = (q_1,q_2)$. The concavity of k^2, (24.2), and $k^2(0) = 0$, imply

(24.3) $k^2(\lambda) \geq \lambda$ for all $\lambda \in [0,q_2]$.

Now let $D := \text{com}\{p \in P(S_\Gamma) : q_1 \leq p_1 \leq \phi_1(\Gamma)\} \in B$. Since $\phi(\Gamma) \in D$ and $\phi(K^2(\Gamma)) \in D$, and further, by (24.3) and Lemma 23.5 : $D \subset S_{K^2(\Gamma)}$, we have by IIA :

(24.4) $\phi(D) = \phi(K^2(\Gamma))$.

On the other hand, $D \subset S_\Gamma$. Hence, by IIA again :

(24.5) $\phi(D) = \phi(\Gamma)$.

Now (24.4), (24.5), and (24.1), are in contradiction. ■

Corollary 24.2. Every Nash solution is risk sensitive on BSC.

Proof. Theorems 24.1 and 11.8. ■

Theorem 24.3. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying PO, STI, and IM. Then ϕ satisfies RS on BSC.

Proof. Let $\Gamma \in$ BSC, and (say) $k^2 \in C^2(\Gamma)$. We show

(24.6) $\phi_1(K^2(\Gamma)) \geqq \phi_1(\Gamma)$.

In view of STI and Lemma 23.5, we may suppose $k^2(h_2(S_\Gamma)) = h_2(S_\Gamma)$. Then the concavity of k^2, and $k^2(0) = 0$, imply

(24.7) $k^2(\lambda) \geqq \lambda$ for all $\lambda \in [0, h_2(S_\Gamma)]$.

In view of Lemma 15.10, Lemma 23.5, (24.7), $h(S_I) = h(S_{K^2(\Gamma)})$, and IM, we obtain $\phi(\Gamma) \leqq \phi(K^2(\Gamma))$, from which (24.6) follows. ■

Corollary 24.4. Every 2-person bargaining solution π^λ ($\lambda \in \Lambda$, see section 15) is risk sensitive on BSC.

Proof Theorems 24.3 and 15.15. ■

Theorem 24.5. Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution satisfying IR, PO, STI, and GIM. Then ϕ satisfies RS on BSC.

Proof. Let $\Gamma \in$ BSC, and (say) $k^2 \in C^2(\Gamma)$. We show

(24.8) $\phi_1(K^2(\Gamma)) \geqq \phi_1(\Gamma)$.

In view of STI, we may suppose $k^2(g_2(S_\Gamma)) = g_2(S_\Gamma)$. Then the concavity of k^2, and $k^2(0) = 0$, imply

(24.9) $k^2(\lambda) \geqq \lambda$ for all $\lambda \in [0, g_2(S_\Gamma)]$.

In view of Lemma 23.5, (24.9), Lemma 16.4, the fact that $g(S_\Gamma) = g(S_{K^2(\Gamma)})$, and GIM, we obtain $\phi(\Gamma) \leqq \phi(K^2(\Gamma))$, from which (24.8) follows. ■

Corollary 24.6. Every 2-person bargaining solution ψ^θ ($\theta \in \Theta$, see section 16) is risk sensitive on BSC.

Proof. Theorems 24.5 and 16.9. ■

Thus, we have shown that most solutions appearing hitherto in this monograph, are risk sensitive on BSC. It can easily be shown that the proportional solution E^p (see section 18) is risk sensitive on BSC if and only if $p = (1,0)$

or p = (0,1). (Note that Theorem 23.13 does not apply here since proportional solutions are not Pareto optimal.)

Next, we remark that all (risk sensitive) solutions discussed in this section have the worse alternative property on BSC, as a consequence of Theorem 23.12.

We conclude with some references to the literature. Kihlstrom, Roth, and Schmeidler (1981) have related results for the (symmetric) Nash solution $N^{\frac{1}{2}}$, the Kalai - Smorodinsky solution KS, and the so-called Super-Additive solution of Perles and Maschler (1981). For the results in this section, see de Koster et al. (1983) and Peters and Tijs (1985a).

25. RISK SENSITIVITY, TWIST SENSITIVITY, AND THE SLICE PROPERTY

In this section, we consider 2-person bargaining solutions defined on B_+ (see Def. 9.10), or restrictions of 2-person bargaining solutions to B_+. This restriction to B_+ is intended to serve convenience of presentation. We further limit attention to underlying bargaining situations in

$$B_+SC = \{\Gamma \in BSC : S_\Gamma \in B_+\}.$$

(This notation is in the same spirit as Def. 10.2.) So we consider only bargaining situations Γ for which every Pareto optimal outcome in the corresponding bargaining game corresponds to a riskless alternative and has both coordinates at least 0. Therefore, and in view of Lemmas 5.6 and 23.5, if $i \in \{1,2\}$ and $k^i \in C^i(\Gamma)$, $S_{K^i(\Gamma)}$ arises from S_Γ by application of a continuous nondecreasing nonconstant concave function, assigning 0 to 0, on the i-th coordinates of the points in $S \cap \mathbb{R}^2_+$, after which the comprehensive hull of the resulting set is taken. For $S \in B_+$, let now $C^i(S)$ denote the family of all continuous nondecreasing nonconstant concave functions $k^i : [0,h_i(S)] \to \mathbb{R}$ with $k^i(0) = 0$. For $k^i \in C^i(S)$, let $K^i(S) \in B_+$ denote the comprehensive hull of the set of points obtained by application of k^i to the i-th coordinates of the points in $S \cap \mathbb{R}^2_+$. The following observation is immediate and will be used throughout this section without our further mentioning it :

(25.1) A 2-person bargaining solution $\phi : B \to \mathbb{R}^2$ is risk sensitive on B_+SC

if and only if $\phi_j(K^i(S)) \geqq \phi_j(S)$ for all $S \in B_+$, $i,j \in \{1,2\}$ with $i \neq j$, $k^i \in C^i(S)$.

We introduce two new properties for a 2-person bargaining solution $\phi : B_+ \to \mathbb{R}^2$, and our main purpose is, to prove a relation between these properties, and risk sensitivity of ϕ on B_+SC. By-products of the main theorem of this section are new proofs of the risk sensitivity (on B_+SC) of the IIA- and IM-solutions of the previous section, since, as we will see, all these solutions are twist sensitive : twist sensitivity is one of the new properties announced above.

Let $S, T \in B_+$, $i \in \{1,2\}$, $\alpha_i \in [0,h_i(S)]$. We say that T is a *favorable twist of* S or S an *unfavorable twist of* T *for player* i *at level* α_i if

(25.2) $x_i > \alpha_i$ for all $x \in T\backslash S$

(25.3) $x_i < \alpha_i$ for all $x \in S\backslash T$.

<u>Definition 25.1.</u> A bargaining solution $\phi : B_+ \to \mathbb{R}^2$ is called *twist sensitive* if for each S and T in B_+ with $\phi(S) \in P(T)$, we have for each $i \in \{1,2\}$:

(25.4) $\phi_i(T) \geqq \phi_i(S)$ if T is a favorable twist of S for player i at level $\phi_i(S)$.

(Twist sensitivity : TS)

<u>Lemma 25.2.</u> Let $\phi : B_+ \to \mathbb{R}^2$ satisfy TS and PO. Let $S, T \in B_+$ with $\phi(S) \in P(T)$, $i \in \{1,2\}$, and suppose that T is an unfavorable twist of S for player i at level $\phi_i(S)$. Then $\phi_i(T) \leqq \phi_i(S)$.

<u>Proof.</u> For all $x \in S\backslash T$, we have $x_i > \phi_i(S)$, hence $x_j < \phi_j(S)$ for $j \neq i$ since $\phi(S) \in P(S)$. For all $x \in T\backslash S$ we have $x_i < \phi_i(S)$, hence $x_j > \phi_j(S)$ for $j \neq i$ since $\phi(S) \in S$. So T is a favorable twist of S for player $j \neq i$ at level $\phi_j(S)$. By TS : $\phi_j(T) \geq \phi_j(S)$; because $\phi(S) \in P(T)$, this implies $\phi_i(T) \leqq \phi_i(S)$. ■

For Pareto optimal solutions, TS is equal to the twisting property Tw (for n=2), introduced in Thomson and Myerson (1980, p.39). In general (for n=2), Tw ⇒ TS.

The following theorem is one of the main results of this section.

<u>Theorem 25.3.</u> Let $\phi : B_+ \to \mathbb{R}^2$ satisfy PO and STI. Let ϕ be twist sensitive.

Then ϕ is risk sensitive on B_+SC.

Proof. Let $S \in B_+$, and, say, $k^2 \in C^2(S)$. We have to prove that

(25.5) $\phi_1(K^2(S)) \geqq \phi_1(S)$.

If $\phi_2(K^2(S)) = 0$ then by PO, we have $\phi_1(K^2(S)) = h_1(K^2(S)) = h_1(S) \geqq \phi_1(S)$, and (25.5) holds. Suppose now that $\phi_2(K^2(S)) > 0$. Since ϕ satisfies STI it is no less of generality to suppose that, for $q = (q_1,q_2) \in P(S)$ with $q_1 = \phi_1(K^2(S))$, we have

(25 6) $k^2(q_2) = q_2$.

By the concavity of k^2 we then have

(25.7) $k^2(\lambda) \geqq \lambda$ for all $\lambda \in [0,q_2]$, $k^2(\lambda) \leqq \lambda$ for all $\lambda \in [q_2,h_2(S)]$.

From (25.6) and (25.7) it follows that S is an unfavorable twist of $K^2(S)$ for player 1 at level $\phi_1(K^2(S))$. From Lemma 25.2 we may conclude that (25.5) holds. ∎

The converse of Theorem 25.3 does not hold as Example 25.7 below shows. We introduce now another property for 2-person bargaining solutions on B_+.

Definition 25.4. A bargaining solution $\phi : B_+ \to \mathbb{R}^2$ is said to have the *slice property* if, for all S, T $\in B_+$ with $h(S) = h(T)$ and $T \subset S$, we have :

(SL_1) $\phi_1(T) \geqq \phi_1(S)$ if $x_2 > \phi_2(S)$ for all $x \in S\backslash T$

(SL_2) $\phi_2(T) \geqq \phi_2(S)$ if $x_1 > \phi_1(S)$ for all $x \in S\backslash T$.

(Slice : SL)

An SL-solution ϕ favours the opponent of a player i when a piece of the outcome set S, preferred by i over $\phi(S)$, is sliced off, the utopia point remaining the same. The slice property resembles the Cutting axiom of Thomson and Myerson (1980), for n=2. The difference (for Pareto optimal solutions) is, that in the Cutting axiom there is no condition on the utopia point, which makes SL weaker than Cutting.

Theorem 25.5. Let $\phi : B_+ \to \mathbb{R}^2$ satisfy PO. Let ϕ be twist sensitive. Then ϕ has the slice property.

Proof. We only prove that (SL_1) holds.

Let S, T $\in B_+$ with $h(S) = h(T)$, $T \subset S$, and $x_2 > \phi_2(S)$ for all $x \in S\backslash T$. We have to show that

(25.8) $\phi_1(T) \geq \phi_1(S)$.

Note that $\phi(S) \in P(T)$ and $x_1 < \phi_1(S)$ for all $x \in S\backslash T$ because $\phi(S) \in P(S)$. Since $T\backslash S = \emptyset$, we may conclude that T is a favorable twist of S for player 1 at level $\phi_1(S)$. So (25.8) follows from (25.4) for i=1. ■

Example 25.8 will show that the converse of Theorem 25.5 does not hold. In the following theorem a characterization of twist/risk sensitivity is given.

<u>Theorem 25.6.</u> Let $\phi : B_+ \to \mathbb{R}^2$ satisfy PO and STI. Then ϕ is twist sensitive if and only if it has the slice property and is risk sensitive on B_+SC.

<u>Proof.</u> (See Fig. 25.1.) The "only if"-part follows from Theorems 25.3 and 25.5. For the "if"-part, let ϕ satisfy SL and RS on B_+SC. Let $S, T \in B_+$ with $\phi(S) \in P(T)$, such that

(25.9) $x_1 > \phi_1(S)$ for all $x \in T\backslash S$

and

(25.10) $x_1 < \phi_1(S)$ for all $x \in S\backslash T$.

Suppose, contrary to what we want to prove, that

(25.11) $\phi_1(T) < \phi_1(S)$.

We derive a contradiction which will complete the proof. Let $k^1 \in C^1(T)$ be defined by : $k^1(t) = t$ if $0 \leqq t \leqq h_1(S)$, $k^1(t) = h_1(S)$ if $h_1(S) \leqq t \leqq h_1(T)$. Then $K^1(T) = \{x \in T : x_1 \leqq h_1(S)\}$, and by RS of ϕ we have $\phi_2(K^1(T)) \geqq \phi_2(T)$, hence, since $\phi(T) \in P(T)$:

(25.12) $\phi_1(K^1(T)) \leqq \phi_1(T)$.

By a similar argument we obtain for $K^2(S) := \{x \in S : x_2 \leqq h_2(T)\}$:

(25.13) $\phi_1(K^2(S)) \geqq \phi_1(S)$.

Let $D := S \cap T \in B_+$. Then $h(D) = (h_1(S), h_2(T)) = h(K^1(T)) = h(K^2(S))$.

If $x \in K^1(T)\backslash D$, then $x \in T\backslash S$, so by (25.9), (25.11), and (25.12), we have $x_1 > \phi_1(S) > \phi_1(T) \geqq \phi_1(K^1(T))$. By ($SL_2$) applied to $D \subset K^1(T)$, we then obtain $\phi_2(D) \geqq \phi_2(K^1(T))$ hence by PO :

(25.14) $\phi_1(D) \leqq \phi_1(K^1(T))$.

By a similar argument, we have $\phi_1(D) \geqq \phi_1(K^2(S))$, which combined with (25.14), (25.12), and (25.13), gives : $\phi_1(S) \leqq \phi_1(T)$. This contradicts (25.11). ■

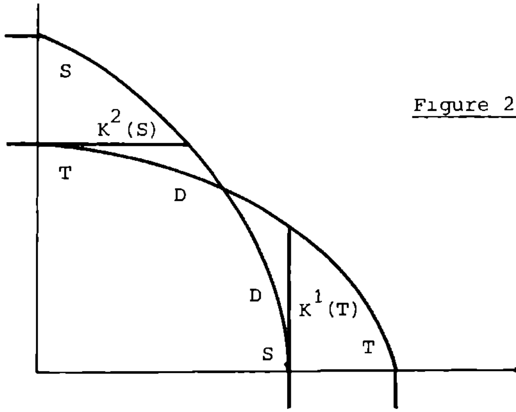


Figure 25.1.

We now discuss some examples of bargaining solutions, with respect to the three properties which are central in this section.

Example 25.7. The so-called Super-Additive solution of Perles and Maschler is risk sensitive (on B_+SC) but not twist sensitive and does not have the slice property. See Counter-Example 7.1, p.189, in Perles and Maschler (1981).

Example 25.8. Let the Pareto optimal, scale transformation invariant solution $\phi : B_+ \to \mathbb{R}^2$ be defined by : for all $S \in B_+$ with $h(S) = (1,1)$, $\phi(S)$ is the point of intersection of $P(S)$ with γ which has maximal second coordinate, where γ is the curve depicted in Fig. 25.2., and described as follows.

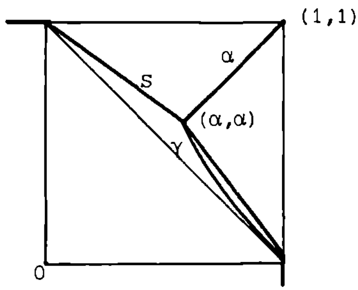


Figure 25.2.

Let $\alpha := \frac{3}{2} - \frac{1}{2}\sqrt{3}$, then between $(1,1)$ and (α,α), $\gamma = \text{conv}\{(1,1),(\alpha,\alpha)\}$, and between (α,α) and $(1,0)$, γ is an arc of the circle $(x_1-2)^2 + (x_2-1)^2 = 2$. By STI, ϕ is uniquely determined for all $S \in B_+$. It is easy to see that ϕ has the slice property. However, ϕ is not twist sensitive. Let $S := \{(1,0),(\alpha,\alpha),(0,1)\}$ and $T := \text{comv}\{(1,0),(0,\alpha(1-\alpha)^{-1})\}$. Then T is an unfavorable twist of S for player 1 at level $\alpha = \phi_1(S)$, but $\phi_1(T) = 1 > \alpha = \phi_1(S)$, so by Lemma 25.2, ϕ is not twist sensitive.

Example 25.9. We give another example of a (continuous) solution which has the slice property but is not twist sensitive (and hence, not risk sensitive). Let this solution $\phi : B_+ \to \mathbb{R}^2$ be defined by : for all $S \in B_+$, $\phi(S)$ maximizes the product $(x_1 + h_1(S))(x_2 + h_2(S))$ on $S \cap \mathbb{R}^2_+$. (So $\phi(S)$ is obtained by calculating the Nash solution with $-h(S)$ as disagreement outcome.) It is easy to see that ϕ satisfies PO, STI, and SL. However, ϕ is not twist sensitive. Let $S := \text{comv}\{(1,0),(0,1)\}$, then $\phi(S) = (\frac{1}{2},\frac{1}{2})$. Let $T := \text{comv}\{(1,0),(\frac{1}{4},\frac{3}{4})\}$, then T is a favorable twist of S for player 1 at level $\phi_1(S) = \frac{1}{2}$. Now $\phi(T) = (\frac{3}{8},\frac{5}{8})$, so $\phi_1(T) < \phi_1(S)$. Hence, ϕ is not twist sensitive.

Example 25.10. Let $\alpha : B_+ \to \mathbb{R}^2$ be the *equal area split solution*, that is, for every $S \in B_+$, $\alpha(S)$ is that point of $P(S)$ such that the area in $S \cap \mathbb{R}^2_+$ lying above the straight line through 0 and $\alpha(S)$ equals half the area of $S \cap \mathbb{R}^2_+$. Then α satisfies PO and STI, is twist sensitive, and consequently is also risk sensitive and slice sensitive.

We conclude this section with two theorems, which, together with Theorem 25.6, can be used to give alternative proofs for Theorems 24.1 and 24.3. Note that for the solutions under consideration (i.e. Nash solutions, and non-symmetric extensions of the Kalai - Smorodinsky solution), the restriction to B_+ is without loss of generality. This would not be the case, however, for the globally individually monotonic solutions in Theorem 24.5.

Theorem 25.11. Let $\phi : B_+ \to \mathbb{R}^2$ be a solution satisfying PO, STI, and IIA. Then ϕ is twist sensitive.

Proof. Let $S, T \in B_+$ with $\phi(S) \in P(T)$ and with T a favorable twist of S for player 1 at level $\phi_1(S)$, i.e.

(25.14) $x_1 > \phi_1(S)$ for all $x \in T\backslash S$

(25.15) $x_1 < \phi_1(S)$ for all $x \in S\backslash T$.

We have to prove that

(25.16) $\phi_1(T) \geq \phi_1(S)$.

Let $D := S \cap T$. Since $D \subset S$ and $\phi(S) \in T$, we have by IIA

(25.17) $\phi(D) = \phi(S)$.

Since $D \subset T$ the IIA-property implies $\phi(D) = \phi(T)$ or $\phi(T) \notin D$.

In the first case, we have $\phi(T) = \phi(S)$ in view of (25.17), so (25.16) holds.

If $\phi(T) \notin D$, then $\phi(T) \in T\backslash S$, and then (25.16) follows from (25.14). ■

The proof of Theorem 25.11 is related to the proof of lemma 5 in Thomson and Myerson (1980). As to the following theorem : Thomson and Myerson (1980, lemma 9 for n=2) show that their property WM (which is somewhat stronger than IM) together with WPO implies Tw. We have :

<u>Theorem 25.12.</u> Let $\phi : B_+ \to \mathbb{R}^2$ be a solution satisfying PO, STI, and IM. Then ϕ is twist sensitive.

<u>Proof.</u> Let $S, T \in B_+$ with $\phi(S) \in P(T)$ and suppose (25.14) and (25.15) hold. We have to show that (25.16) holds. Let $D := S \cap T$. Since $D \subset S$ and $h_1(D) = h_1(S)$ by (25.15), we have by IM that $\phi_2(S) \geq \phi_2(D)$. Since $\phi(S) \in D$ and $\phi(D) \in P(D)$, $\phi_2(S) \geq \phi_2(D)$ implies that

(25.18) $\phi_1(S) \leq \phi_1(D)$.

From $D \subset T$, $h_2(D) = h_2(T)$ and IM we may conclude $\phi_1(D) \leq \phi_1(T)$ which, together with (25.18), implies (25.16). ■

The results of this section were published in Tijs and Peters (1985).

26. RISK SENSITIVITY AND RISKY PARETO OPTIMAL OUTCOMES

We consider bargaining situations $\Gamma = \langle A,\bar{a},u^1,u^2 \rangle$ where not necessarily every $x \in P(S_\Gamma)$ is *riskless*, i.e. x may be *risky*, which means : there exists no $a \in A$ with $x = (u^1(a),u^2(a))$. We have seen (Theorem 23.6) that there exists no weakly Pareto optimal, individually rational bargaining solution $\phi : B \to \mathbb{R}^2$ that is risk sensitive (i.e. risk sensitive on BS). There are, however, some results on risk properties for specific cases : see Roth and

Rothblum (1982) for the (symmetric) Nash solution, and Peters (1981) for non-symmetric Nash solutions, and the Kalai - Rosenthal solution (cf. section 16). We do not describe these results in detail, because that would require the introduction of quite many new notations which will not be used any further.

The proof of the "impossibility" Theorem 23.6 indicates that the meant impossibility (see above) is due to the fact that an alternative which does not have nonnegative utilities for both players, may still influence the solution outcome. Indeed, if we restrict attention to bargaining games in B_+ and bargaining situations in B_+S, then we have the following result.

<u>Theorem 26.1.</u> Let the bargaining solution $\phi : B_{(+)} \to \mathbb{R}^2$ be risk sensitive on B_+SC (cf. the first paragraph of section 25). Then ϕ is risk sensitive on B_+S.

The proof of Theorem 26.1 will be postponed for the moment. It first appeared in Wakker, Peters, and van Riel (1985). Some consequences are collected in the following corollary.

<u>Corollary 26.2.</u> Let $\phi : B_+ \to \mathbb{R}^2$ be a bargaining solution satisfying PO and STI. Then :

(i) If ϕ satisfies IR and IIA, then it satisfies RS and WA on B_+S.

(ii) If ϕ satisfies IM, then it satisfies RS and WA on B_+S.

<u>Proof.</u> Theorems 24.1 and 24.3 (or 25.11, 25.12, 25.3), Lemma 23.7, Theorem 26.1. ■

Theorem 26.1 enables us to establish the risk sensitivity of bargaining solutions on a subclass of the class of all bargaining situations, containing also situations with risky Pareto optimal outcomes in the corresponding bargaining games, from the risk sensitivity of these solutions on B_+SC. For the proof of Theorem 26.1, we need the following lemma. Roughly, this lemma says that, if we apply a continuous nondecreasing concave function on the second coordinates of the points of a nonempty compact subset of $\mathbb{R}^2$ with only Pareto optimal points, then we can extend this function to a function with the same properties defined on the convex hull of the set of second coordinates, such that the Pareto optimal surface of the convex hull of the original set is mapped into the (weak) Pareto optimal surface of the convex hull of the new set. Formally :

Lemma 26.3. Let $X \subset \mathbb{R}^2$ be nonempty and compact, with $X = P(X)$. Let $Y := \mathrm{conv}(X)$. Let $k : \{x_2 \in \mathbb{R} : (x_1,x_2) \in X$ for some $x_1 \in \mathbb{R}\} \to \mathbb{R}$ be continuous, nondecreasing and concave, such that $X' \subset P(Y')$ where $X' := \{(x_1,k(x_2)) : (x_1,x_2) \in X\}$ and $Y' := \mathrm{conv}(X')$. Then $f^2_{Y'}(\lambda) = \bar{k} \circ f^2_Y(\lambda)$ with $\bar{k} : [\underline{p}^Y_2, \bar{p}^Y_2] \to \mathbb{R}$ continuous, nondecreasing, and concave, for all λ in the domain of $f^2_{Y'}(\lambda)$.
[Here f^2_Y, $f^2_{Y'}$ are the Pareto functions, defined analogous to the functions f^1_S in Def. 9.12, and $\underline{p}^Y$ and $\bar{p}^Y$ denote the right and left endpoints of $P(Y)$, respectively.]

We first give an example to illustrate this lemma, the proof of which is postponed.

Example 26.4. Let $I = \langle A,\bar{a},u^1,u^2\rangle \in BS$ with $A = \{\bar{a},a^1,a^2,a^3,a^4\}$, $u^1(\bar{a}) = u^2(\bar{a}) = 0$, $u^1(a^1) = u^2(a^2) = 1$, $u^1(a^2) = u^2(a^1) = 0$, $u^1(a^3) = \frac{1}{2}$, $u^2(a^3) = \frac{3}{8}$, $u^1(a^4) = \frac{3}{4}$, $u^2(a^4) = \frac{1}{32}$. Let $k^2 \in C^2(\Gamma)$ be defined by $k^2 : \lambda \mapsto \sqrt{\lambda}$.
With notations as in Lemma 26.3, we let $X := \{a^1,a^2,a^3\}$, $k := k^2$. Then the conditions of the lemma are fulfilled. We have $\bar{k} : [0,1] \to \mathbb{R}$ defined by $\bar{k}(\lambda) = \frac{1}{2}\lambda\sqrt{6}$ if $\lambda \in [0,\frac{1}{2}]$, and $\bar{k}(\lambda) = (2 - \frac{1}{2}\sqrt{6})\lambda - 1 + \frac{1}{2}\sqrt{6}$ if $\lambda \in [\frac{1}{2},1]$.
Now let $\Gamma' := \langle\{(0,0)\} \cup \mathrm{conv}\{(1,0),(0,1)\},(0,0),\pi^1,\pi^2\rangle$, so Γ' is the trivial bargaining situation corresponding to S_Γ (cf. Example 9.6). In particular, we have $S_\Gamma = S_{\Gamma'}$ and $S_{K^2(\Gamma)} = S_{\bar{K}(\Gamma')}$ where $\bar{K}(\Gamma')$ is obtained from Γ' by substitution of $\bar{k} \circ \pi^2$ for π^2. The advantage of replacing I and $K^2(\Gamma)$ by Γ' and $\bar{K}(\Gamma')$, respectively, is that Γ', $\bar{K}(\Gamma') \in BSC$ whereas I, $K^2(\Gamma) \notin BSC$. This idea will be applied in the proof of Theorem 26.1.

Proof of Theorem 26.1. We suppose that $\phi : B_+ \to \mathbb{R}^2$ is risk sensitive on B_+SC. Let now $\Gamma = \langle A,\bar{a},u^1,u^2\rangle \in B_+S$, and, say, $k^2 \in C^2(\Gamma)$. We have to prove

(26.1) $\phi_1(S_{K^2(\Gamma)}) \geq \phi_1(S_\Gamma)$.

We apply Lemma 26.3 with $X := \{(u^1(a),u^2(a)) : a \in A, (u^1(a),k^2 \circ u^2(a)) \in P(S_{K^2(\Gamma)})\}$. Then X and $k := k^2$ satisfy the conditions of Lemma 26.3, so, with notations as in that lemma, $f^2_{Y'}(\lambda) = \bar{k} \circ f^2_Y(\lambda)$ with $\bar{k} : [\underline{p}^Y_2,\bar{p}^Y_2] \to \mathbb{R}$ continuous, nondecreasing, and concave, for all λ in its domain. If $0 \notin [\underline{p}^Y_2,\bar{p}^Y_2]$, then we further extend $\bar{k}$ to a function ℓ which assigns 0 to 0, where ℓ is well-defined and inherits the properties of $\bar{k}$ since $\bar{k}$ extends $k = k^2$. We write $\bar{k}$ instead of ℓ. Now let Γ' be the trivial bargaining situation corresponding to S_Γ. Note that $\Gamma' \in B_+SC$, and that $\bar{k} \in C^2(\Gamma')$. So by RS of ϕ on B_+SC we have

(26.2) $\phi_1(S_{\bar{K}(\Gamma')}) \geq \phi_1(S_{\Gamma'}) = \phi_1(S_\Gamma)$.

Since, further, $S_{K^2(\Gamma)} = S_{\bar{K}(\Gamma)}$, we may conclude (26.1) from (26.2). ∎

The remainder of this section is devoted to the proof of Lemma 26.3. In this proof, we use the following auxiliary results. For the definition of concavity of a function on a nonconvex domain, see Def. 5.3.

Lemma 26.5. Let $V \subset \mathbb{R}$, $k : V \to \mathbb{R}$. Equivalent are :

(i) k is concave.

(ii) $k(\alpha x + (1-\alpha)y) \geq \alpha k(x) + (1-\alpha)k(y)$ for all $x, y \in V$ and $\alpha \in [0,1]$ with $\alpha x + (1-\alpha)y \in V$.

Proof. (i) $\Rightarrow$ (ii) is by Definition 5.3. Now suppose (ii) holds. Let $\ell \in \mathbb{N}$, $x, x_1, x_2, \ldots, x_\ell \in V$ and $\lambda_1, \lambda_2, \ldots, \lambda_\ell > 0$ with $\Sigma_{i=1}^{\ell}\lambda_i = 1$ and $x = \Sigma_{i=1}^{\ell}\lambda_i x_i$. We have to prove :

(26.3) $k(x) \geq \Sigma_{i=1}^{\ell}\lambda_i k(x_i)$.

The proof is by induction on ℓ. Suppose (26.3) holds for all $\ell < n$. We shall now prove (26.3) for $\ell = n > 2$. W.l.o.g. we may suppose that $0 < x_1 < x_2 < \ldots < x_n$. Let $1 < j \leq n$ and $\alpha \in [0,1]$ be such that $x = \alpha x_{j-1} + (1-\alpha)x_j$. Since $\alpha + (1-\alpha) = 1 > \lambda_{j-1} + \lambda_j$, we have $\alpha > \lambda_{j-1}$, or $1 - \alpha > \lambda_j$. W.l.o.g. say $1 - \alpha > \lambda_j$. By rearranging terms in $x = \alpha x_{j-1} + (1-\alpha)x_j = \Sigma_{i=1}^{n}\lambda_i x_i$ and dividing by $1 - \alpha - \lambda_j$, we obtain :

$$(26.4) \quad x_j = \frac{\Sigma_{i\neq j-1,j}\lambda_i x_i + (\lambda_{j-1} - \alpha)x_{j-1}}{1 - \alpha - \lambda_j}$$

which means that x_j is a convex combination of n-1 elements of V, so by induction :

(26.5) $(1 - \alpha - \lambda_j)\, k(x_j) \geq \Sigma_{i\neq j-1,j}\lambda_i k(x_i) + (\lambda_{j-1} - \alpha)\, k(x_{j-1})$.

By (26.5) and (ii), $k(x) \geq \alpha k(x_{j-1}) + (1-\alpha)\, k(x_j) \geq \Sigma_{i=1}^{n}\lambda_i k(x_i)$, which is (26.3) for $\ell = n$. ∎

Lemma 26.6. Let X, k, Y, Y' be as in Lemma 26.3, with the conditions of that lemma fulfilled. Let $w = (w_1, w_2) \in X \setminus P(Y)$, let $\bar{w}_2 > w_2$ be the second coordinate of the point in $P(Y)$ with first coordinate w_1, and let $k_{\{w\}} : \{x_2 \in \mathbb{R} : x_2 \neq w_2, (x_1, x_2) \in X \text{ for some } x_1 \in \mathbb{R}\} \cup \{\bar{w}_2\} \to \mathbb{R}$ be defined by $k_{\{w\}}(\bar{w}_2) = k(w_2)$ and $k_{\{w\}}(x_2) = k(x_2)$ if $x_2 \neq \bar{w}_2$. Then $k_{\{w\}}$ is concave.

Proof. For convenience, we write $\bar{k}$ instead of $k_{\{w\}}$. Let x_2, y_2, z_2 be in the domain of $\bar{k}$, with $x_2 > z_2$ and $y_2 = \alpha x_2 + (1-\alpha) z_2$ for some $0 < \alpha < 1$. In view of Lemma 26.5 it is sufficient to show :

$$(26.6) \quad \bar{k}(y_2) \geq \alpha\bar{k}(x_2) + (1-\alpha)\bar{k}(z_2).$$

If $\bar{w}_2 \neq x_2, y_2, z_2$, then (26.6) follows from the concavity of k. Otherwise, there remain two cases : (i) $\bar{w}_2 = z_2$ (ii) $\bar{w}_2 = y_2$. (The case $\bar{w}_2 = x_2$ is analogous to case (i).)

(i) Let $\bar{w}_2 = z_2$. We write $y_2 = \beta x_2 + (1-\beta) w_2$ with $1 > \beta > \alpha$. Then $\bar{k}(y_2) = k(y_2) \geq \beta k(x_2) + (1-\beta) k(w_2) \geq \alpha\bar{k}(x_2) + (1-\alpha)\bar{k}(z_2)$, where the last inequality follows from the nondecreasingness of k.

(ii) Let $\bar{w}_2 = y_2$. There are $s, t \in X \cap P(Y)$ with $x_2 \geq s_2 > \bar{w}_2 > t_2 \geq z_2$ such that $y = (w_1, \bar{w}_2) = \delta s + (1-\delta)t$ for some $0 < \delta < 1$. Concavity of k implies $\delta k(s_2) + (1-\delta) k(t_2) \geq \alpha k(x_2) + (1-\alpha) k(z_2)$. Hence $\bar{k}(y_2) \geq \alpha k(x_2) + (1-\alpha) k(z_2) = \alpha\bar{k}(x_2) + (1-\alpha)\bar{k}(z_2)$ since otherwise $(y_1, \bar{k}(y_2)) = (y_1, k(w_2)) = (w_1, k(w_2)) \in X' \subset P(Y')$ would be contradicted. ∎

Lemma 26.7. With notations and conditions as in Lemma 26.6, let $w^1, w^2, \ldots, w^\ell \in X \setminus P(Y)$, and let $k_{\{w^1,\ldots,w^\ell\}} : \{x_2 \in \mathbb{R} : x_2 \neq w_2^1, \ldots, w_2^\ell, (x_1, x_2) \in X \text{ for some } x_1 \in \mathbb{R}\} \cup \{\bar{w}_2^1, \ldots, \bar{w}_2^\ell\} \to \mathbb{R}$ be defined by $k_{\{w^1,\ldots,w^\ell\}}(\bar{w}_2^i) = k(w_2^i)$ for $i=1,2,\ldots,\ell$, and $k_{\{w^1,\ldots,w^\ell\}}(x_2) = k(x_2)$ otherwise. Here, for $i=1,2,\ldots,\ell$, $\bar{w}_2^i$ is again the second coordinate of the point in $P(Y)$ with first coordinate w_1^i. Then $k_{\{w^1,\ldots,w^\ell\}}$ is concave.

Proof. First, apply Lemma 26.6 to obtain the concavity of $k_{\{w^1\}}$. Note that $k_{\{w^1\}}$ is also nondecreasing. Then apply Lemma 26.6 again to obtain the concavity of $(k_{\{w^1\}})_{\{w^2\}}$, etc., until, finally, the concavity of

$(\ldots((k_{\{w^1\}})_{\{w^2\}})\ldots)_{\{w^\ell\}}$ results. Then note that this last function is equal to $k_{\{w^1,w^2,\ldots,w^\ell\}}$. ■

Proof of Lemma 26.3. For every $x = (x_1,x_2) \in X$, denote by $\bar{x} = (\bar{x}_1,\bar{x}_2)$ the point in $P(Y)$ with first coordinate equal to x_1. Define the function $\bar{k} : W := \{\bar{x}_2 \in \mathbb{R} : x = (x_1,x_2) \in X\} \to \mathbb{R}$ by $\bar{k}(\bar{x}_2) := k(x_2)$ for every $\bar{x}_2 \in W$. Then $\bar{k}$ is well-defined since $X = P(X)$, and $\bar{k}$ is nondecreasing since k is nondecreasing. For concavity it is in view of Lemma 26.5 sufficient to show, for $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2 \in W$ with $\bar{x}_2 > \bar{z}_2$ and $\bar{y}_2 = \alpha\bar{x}_2 + (1-\alpha)\bar{z}_2$ for some $0 < \alpha < 1$, that $\bar{k}(\bar{y}_2) \geq \alpha\bar{k}(\bar{x}_2) + (1-\alpha)\bar{k}(\bar{z}_2)$. In order to prove this, we may suppose that $X\backslash P(Y) \subset \{x,y,z\}$; but then the inequality follows by application of Lemma 26.7. Now let $\lambda \in [\underline{p}_2^Y,\bar{p}_Y^2]$ such that $\lambda \notin W$. Then $\bar{\lambda} := \min\{\bar{x}_2 \in W : \bar{x}_2 > \lambda\}$ and $\underline{\lambda} := \max\{\bar{x}_2 \in W : \bar{x}_2 < \lambda\}$ exist since X is compact, and $\lambda = \beta\bar{\lambda} + (1-\beta)\underline{\lambda}$ for some $0 < \beta < 1$. Put $\bar{k}(\lambda) := \beta\bar{k}(\bar{\lambda}) + (1-\beta)\bar{k}(\underline{\lambda})$. By this definition, $\bar{k}$ is extended to a nondecreasing concave function on $[\underline{p}_2^Y,\bar{p}_2^Y]$; hence it is continuous on $(\underline{p}_2^Y,\bar{p}_2^Y]$. Continuity in $\underline{p}_2^Y$ follows from the continuity of k in $\underline{p}_2^Y$. ■

27. STRATEGIC RISK AVERSION

Several authors (Kannai (1977); Crawford and Varian (1979); and Sobel (1981)) consider the question (or questions related to it) whether (under certain conditions) it is advantageous for a bargainer to pretend to be less risk averse. We will also briefly address this question in the present section, as a special case of a more general observation which involves the b-monotonicity property for 2-person bargaining solutions (cf. Def. 13.12).

We consider the class $\mathcal{D}$ of *division problems* $D = \langle U^1,U^2\rangle$ where :

(27.1) $U^1 : [0,1] \to \mathbb{R}_+$ is a nondecreasing continuous concave function with $U^1(0) = 0$ and $U^1(1) > 0$.

(27.2) $U^2 : [0,1] \to \mathbb{R}_+$ is a nonincreasing continuous concave function with $U^2(0) > 0$ and $U^2(1) = 0$.

As interpretation of a $D = \langle U^1, U^2 \rangle \in \mathcal{D}$, we say that for every $t \in [0,1]$, $U^1(t)$ is the utility for bargainer 1 of receiving $100\,t$ percent of one unit of a certain good, and $U^2(t)$ is the utility to bargainer 2 of receiving $100(1-t)$ percent of one unit of the same good. We call $S(D) := \mathrm{com}\{(U^1(t), U^2(t)) : 0 \leq t \leq 1\}$ the *bargaining game correponding to* D. Note that $S(D) \in B_+$ and $P(S(D)) = \{(U^1(t), U^2(t)) : 0 \leq t \leq 1\}$. Also, for every $S \in B_+$ we can find a $D \in \mathcal{D}$ with $S = S(D)$.
(Proof : similar to the proof of Lemma 13.9.)

Remark 27.1. If $A = \langle u^1, u^2 \rangle$ is an allocation problem, then $\langle U^1, U^2 \rangle$ (see Def. 13.4) is a division problem.

With every division problem $D = \langle U^1, U^2 \rangle$ we can associate a 2-person bargaining situation

$$\Gamma(D) = \langle \{(t,1-t) : 0 \leq t \leq 1\} \cup \{(0,0)\}, (0,0), v^1, v^2 \rangle$$

with $v^1(t,1-t) = U^1(t)$, $v^2(t,1-t) = U^2(t)$, $v^1(0,0) = v^2(0,0) = 0$, for every $0 \leq t \leq 1$. Of course $S_{\Gamma(D)} = S(D)$, and $\Gamma(D) \in BSC$. One difference between $\Gamma(D)$ and D is the existence, by definition, of lotteries on the set of alternatives in $\Gamma(D)$.

We assume that any $S \in B_+$ occuring in this section corresponds to $\Gamma(D)$ for some $D \in \mathcal{D}$. The following, and main, result of this section involves the b-monotonicity property of 2-person bargaining solutions. (Recall that section 13 provides an interpretation of this property with the aid of competitive equilibria of simple markets, for the case $b > 0$; in particular, see Remark 13.15.)

Observation 27.2. Let $b \in \mathbb{R}^2_+$ with $b_1 + b_2 = 1$, and let $\phi : B_+ \to \mathbb{R}^2$ be a weakly Pareto optimal and b-monotonic bargaining solution.
Let $D = \langle V^1, U^2 \rangle \in \mathcal{D}$ and $D' = \langle U^1, V^2 \rangle \in \mathcal{D}$ with V^1 and V^2 affine functions. Then, for $(t,1-t) \in \mathrm{alt}(\phi, \Gamma(D))$ and $(t', 1-t') \in \mathrm{alt}(\phi, \Gamma(D'))$, we have $t \geq b_1$ and $1-t' \geq b_2$.
Proof. Straightforward. ∎

So, an affine utility function V^1 guarantees player 1 the utility of receiving at least b_1, under the conditions of Observation 27.2, and similarly, player 2 can secure for himself the utility of receiving at least b_2, all provided a division of the good as in the Observation actually takes place. If, however, the final agreement in the bargaining game is effectuated by some non-trivial lottery, then an affine utility function may not be desirable, e.g. if a bargainer's true utility function is not affine : we will not go into further details, but refer to Example 23.4 and the paragraph following that example. As already indicated a few lines ago, Observation 27.2 may play a role if the players can report any utility functions as long as these satisfy (27.1) and (27.2); further,many bargaining solutions in this monograph satisfy its conditions, as the following theorem and corollary show.

<u>Theorem 27.3.</u> Let $\phi : B_+ \to \mathbb{R}^2$ be an individually rational, weakly Pareto optimal, 2-person bargaining solution. Suppose further that ϕ is risk sensitive on B_+SC and that ϕ has the slice property (cf. section 25, especially (25.1)). Let $b = \phi(\Delta)$. Then ϕ is b-monotonic.

<u>Proof.</u> Let $S \in B_+$. We want to prove that $\phi(S) \geqq bh(S)$. First note that for any $a \in \mathbb{R}^2_{++}$ we have $\phi(a\Delta) = a\phi(\Delta) = ab$; this can be proved in the same way as Theorem 23.13, mainly since $W(\Delta) \cap \mathbb{R}^2_+ = P(\Delta)$; so the restriction of ϕ to games $a\Delta$ is Pareto optimal. For notational simplicity, we suppose further that $h(S) = (1,1)$. We assume that $b_1 < 1$ (the case $b_1 = 1$ is analogous to the case $b_2 = 1$). Now suppose, contrary to what we want to prove, that $\phi_2(S) < b_2 = \phi_2(\Delta)$. Let T be the game in B_+ consisting of all points of S except those strictly above the straight line through (0,1) and $\phi(S)$. By SL of ϕ and the definition of T, we have $\phi_2(T) \leqq \phi_2(S)$, so $\phi_2(T) < b_2$. Since $\phi_2(S) < 1$, the function $k^1 : \lambda \mapsto f^1_T(1-\lambda)$ for all $0 \leqq \lambda \leqq 1$, is an element of $C^1(\Delta)$. Because $T = K^1(\Delta)$, we have by RS of ϕ, that $\phi_2(T) \geqq \phi_2(\Delta) = b_2$. So we have a contradiction and conclude that $\phi_2(S) \geqq b_2$. Similarly, one shows that $\phi_1(S) \geqq b_1$. ■

<u>Corollary 27.4.</u> Every Pareto optimal, scale transformation invariant, and twist sensitive solution ϕ on B_+ is $\phi(\Delta)$-monotonic. Every $\phi \in N$ and every $\phi \in \{\pi^\lambda : \lambda \in \Lambda\}$ is $\phi(\Delta)$-monotonic.

<u>Proof.</u> Theorems 25.6 and 27.3, 25.11 and 25.12. ■

We conclude this section with an example which shows that the slice property in Theorem 27.3 cannot be dispensed with.

<u>Example 27.5.</u> Let $\phi : B_+ \to \mathbb{R}^2$ be defined as follows :

$$\phi(S) := \begin{cases} N(S) \text{ if } (0,h_2(S)) \in P(S) \\ D^1(S) \text{ if } (0,h_2(S)) \notin P(S) \end{cases}$$

Then ϕ is (weakly) Pareto optimal and individually rational, and risk sensitive on B_+SC; ϕ does not have the slice property, and is not symmetrically monotonic. Verification of these statements is left to the reader.

CHAPTER 9

PROPERTIES OF n-PERSON BARGAINING SOLUTIONS

In a so-called *n-person game without sidepayments*, each nonempty *coalition* $M \subset N$ (including the all-player coalition N) can achieve any point in some subset $V(M)$ of $\mathbb{R}^M$ (which denotes the Cartesian product of $|M|$ copies of $\mathbb{R}$ indexed by the names of the players in M) by cooperation; in particular, we assume the utility functions of the players to be normalized such that $V(\{i\}) = (-\infty, 0]$ for every $i \in N$. If for every $M \subsetneq N$ we have $V(M) = \{x \in \mathbb{R}^M : x \leqq 0\}$, and if $V(N)$ satisfies the conditions in Def. 9.1, then the game is an n-person ("pure") bargaining game, in which only the disagreement outcome 0 and the grand coalition N play a role. If $n=2$, then every game without sidepayments is a bargaining game : this is one of the main reasons why the case $n=2$ has attracted considerably more attention than the general case of n-person bargaining games has.

Until now, there have been some articles also on characterizations of "values" for n-person games without sidepayments; we mention Hart (1985), Aumann (1985), Kalai and Samet (1985). Here, we confine attention to n-person bargaining games and solutions. Of all the results for 2-person bargaining solutions obtained so far, we only generalize a few ones to the n-person case. In section 28 we consider the n-person independence of irrelevant alternatives property, and obtain an extension of Theorem 11.8 with the aid of an additional property called "consistency". In section 29, we consider the n-person individual monotonicity property and obtain an n-person version of Theorem 15.15. Finally, in section 30, we consider risk properties of n-person bargaining solutions : in particular, we introduce a new property called "risk profit opportunity".

28. INDEPENDENCE OF IRRELEVANT ALTERNATIVES

In section 11, we have characterized the family $\mathcal{N} = \{N^t : 0 \leqq t \leqq 1\}$ of nonsymmetric Nash solutions by the four properties : IR, PO, STI, and IIA. The main question we ask in this section is : how can this result be generalized to n-person bargaining solutions with $n > 2$?

For completeness' sake, we now first give the definition of IIA for the general case $n \geq 2$; for IR, PO, and STI, see section 10.

Definition 28.1. An n-person bargaining solution $\phi : B^n \to \mathbb{R}^n$ is called *independent of irrelevant alternatives* if $\phi(S) = \phi(T)$ for all S and T in B^n with $S \subset T$ and $\phi(T) \in S$.
(Independence of irrelevant alternatives : IIA)

So Definition 28.1 equals Definition 11.1 with B^n instead of B. We now give an example of a 4-person bargaining solution which satisfies IR, PO, STI, and IIA.

Example 28.2. Let $\phi : B^4 \to \mathbb{R}^4$ be defined as follows. For $S \in B^4$, let $\phi_4(S) = \max\{x_4 : x \in S, x \geq 0\}$. Let $Z = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1,x_2,x_3,\phi_4(S)) \in S\}$ If $z > 0$ for some $z \in Z$, then let $\phi(S)$ be such that $(\phi_1(S),\phi_2(S),\phi_3(S))$ maximizes the product $x_1x_2x_3$ on $Z \cap \mathbb{R}^3_+$. Otherwise, let $(\phi_1(S),\phi_2(S),\phi_3(S))$ be the lexicographical maximum of $Z \cap \mathbb{R}^2_+$. We leave it to the reader to verify that ϕ satisfies IR, PO, STI, and IIA.

The solution ϕ of Example 28.2 is further explored in the next example.

Example 28.3. Let S and T in B^4 be defined by
$S = \text{comv}\{(1,0,0,1),(0,1,0,1),(0,0,1,1)\}$ and
$T = \text{comv}\{(1,0,0,0),(0,\frac{2}{3},0,1),(0,0,\frac{2}{3},1)\}$. Let $\phi : B^4 \to \mathbb{R}^4$ be as in Example 28.2. Then $\phi(S) = (\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$, $\phi(T) = (0,\frac{2}{3},0,1)$. Now, if we fix in the game S player 1's utility at $\frac{1}{3}$, then for the other players there remains $\text{com}\{(\frac{2}{3}-\alpha,\alpha,1) : 0 \leq \alpha \leq \frac{2}{3}\} = \{(x_2,x_3,x_4) \in \mathbb{R}^3 : (\phi_1(S),x_2,x_3,x_4) \in S\}$. Further, also $\{(x_2,x_3,x_4) \in \mathbb{R}^3 : (\phi_1(T),x_2,x_3,x_4) \in T\} = \text{com}\{(\frac{2}{3}-\alpha,\alpha,1) : 0 \leq \alpha \leq \frac{2}{3}\}$, but nevertheless we have $(\phi_2(S),\phi_3(S),\phi_4(S)) = (\frac{1}{3},\frac{1}{3},1) \neq (\frac{2}{3},0,1) = (\phi_2(T),\phi_3(T),\phi_4(T))$.

Example 28.3 shows that the solution of Example 28.2 violates the property described in Definition 28.4 below. In order to state this property, we need some additional notations : for a bargaining solution $\phi : B^n \to \mathbb{R}^n$,

$S \in B^n$, $x \in \mathbb{R}^n$, and $\emptyset \neq M \underset{\neq}{\subset} N = \{1,2,\ldots,n\}$, we denote by $x_{N\setminus M}$ the vector obtained from x by deleting the coordinates with indexes in M, and by

$$O_{-M}(\phi,S) := \{x_{N\setminus M} : x \in S \text{ with } x_M = \phi_M(S)\}$$

the *opportunity set for the bargainers* $i \notin M$ *with respect to* ϕ *and* S. The opportunity set $O_{-M}(\phi,S)$ consists of those utility $(n - |M|)$- tuples, available for the collective $N\setminus M$, if the bargainers j in M receive $\phi_j(S)$. We write $O_{-j}(\phi,S)$ instead of $O_{-\{j\}}(\phi,S)$.

Definition 28.4. An n-person bargaining solution $\phi : B^n \to \mathbb{R}^n$ is called *consistent* if for all S and T in B^n and every $\emptyset \neq M \subsetneq N$ we have : $O_{-M}(\phi,S) = O_{-M}(\phi,T)$ implies $\phi_i(S) = \phi_i(T)$ for all $i \notin M$.
(Consistency : CONS)

We shall restrict attention to bargaining solutions which have, besides IR, WPO, STI, and IIA, the consistency property : so we exclude a solution like the one in Example 28.2.

As before, we let N denote the set of n players $\{1,2,\ldots,n\}$. If we are dealing with a non-empty subset $M \subset N$ with $m = |M|$, it will often be convenient to let our notations express that we are considering this specific subset. For instance, we denote by B^M the family of m-person bargaining games where the players are those of M, and keep the same names, i.e. numbers as in N. Similarly, $\mathbb{R}^M$ denotes the Cartesian product of m copies of $\mathbb{R}$, indexed by the names of the players in M.

We proceed by giving a definition, in which we partition the player set N into a hierarchy of subsets, with each player in each subset having a weight which determines his power with respect to the other players in that subset.

Definition 28.5. A *weighted hierarchy* H *of* N is an ordered (k+1)-tuple of the form $H = \langle N^1, N^2, \ldots, N^k, \omega \rangle$ where $(N^1, N^2, \ldots, N^k)$ is a partition of N (with $1 \leqq k \leqq n$) and $\omega \in \mathbb{R}^n_{++}$ with $\Sigma_{i \in N^\ell} \omega_i = 1$ for every $\ell = 1,2,\ldots,k$. We call N^ℓ the *ℓ-th class of* H. By H^N we denote the family of all weighted hierarchies of N.

We will associate with each weighted hierarchy $H \in H^N$ an n-person bargaining solution ϕ^H, by lexicographically maximizing "Nash products" in a game S according to the partition and weights in H. Before we can give a formal definition, we need some more definitions and notations, and a lemma.

Definition 28.6. (i) For $\emptyset \neq L \subset M \subset N$ and $V \subset \mathbb{R}^M$, we call V *nondegenerate for* L if $x \in V$ exists with $x_L > 0$, where, as before, x_L is obtained from x by deleting the coordinates with indexes in $M \backslash L$. If V is not nondegenerate for L, then we call V *degenerate for* L. If V is degenerate for every non-empty subset of L, then we call V *strongly degenerate for* L.
(ii) For $V \subset W \subset \mathbb{R}^n$ and a function $f : W \to \mathbb{R}$, we use the somewhat loose notation: $\operatorname{argmax}\{f(x) : x \in V\} := \{x \in V : f(x) \geq f(y) \text{ for all } y \in V\}$.
(iii) Let $H = \langle N^1, N^2, \ldots, N^k, \omega \rangle$ be a weighted hierarchy of N, and let $S \in B^n$. For $\ell = 0,1,\ldots,k$, we define the sets S^ℓ as follows.

$S^0 := P(S) \cap \mathbb{R}^n_+$, $S^1 := \operatorname{argmax}\{\prod x_i^{\omega_i} : i \in N^1,\ x \in S^0\}$,

$$S^2 := \begin{cases} \operatorname{argmax}\{\prod x_i^{\omega_i} : i \in N^2 \text{ with } S^1 \text{ nondegenerate for } i,\ x \in S^1\}, \\ \quad \text{if } S^1 \text{ is not strongly degenerate for } N^2 \\ S^1, \text{ otherwise} \end{cases}$$

$\vdots$

$$S^\ell := \begin{cases} \operatorname{argmax}\{\prod x_i^{\omega_i} : i \in N^\ell \text{ with } S^{\ell-1} \text{ nondegenerate for } i,\ x \in S^{\ell-1}\}, \\ \quad \text{if } S^{\ell-1} \text{ is not strongly degenerate for } N^\ell \\ S^{\ell-1}, \text{ otherwise.} \end{cases}$$

$\vdots$

$$S^k := \begin{cases} \operatorname{argmax}\{\prod x_i^{\omega_i} : i \in N^k \text{ with } S^{k-1} \text{ nondegenerate for } i,\ x \in S^{k-1}\}, \\ \quad \text{if } S^{k-1} \text{ is not strongly degenerate for } N^k \\ S^{k-1}, \text{ otherwise.} \end{cases}$$

Definition 28.7. Let $\emptyset \neq L \subset M \subset N$. By $e^L \in \mathbb{R}^M$ we denote the vector with $e^L_i = 1$ if $i \in L$, $e^L_i = 0$ otherwise. In particular, we write e^i instead of $e^{\{i\}}$, if $L = \{i\}$. For $x \in \mathbb{R}^L$, we denote by $O^M(x) \in \mathbb{R}^M$ the vector with $O^M(x)_i = x_i$ if $i \in L$, $O^M(x)_i = 0$ otherwise; and by $E^M(x) \in \mathbb{R}^M$ the vector with

$E^M(x)_i = x_i$ if $i \in L$, $E^M(x)_i = 1$ otherwise. For $S \subset \mathbb{R}^L$, we denote $O^M(S) := \{O^M(x) : x \in S\}$ and $E^M(S) := \{E^M(x) : x \in S\}$.

Lemma 28.8. Let $H = \langle N^1, N^2, \ldots, N^k, \omega \rangle \in \mathcal{H}^N$. Let S^k be the set described in Def. 28.6 (iii). Then :

(i) $|S^k| = 1$ for every $S \in B^n$.

(ii) The bargaining solution, denoted ϕ^H, which assigns to every $S \in B^n$ the unique element in S^k, satisfies IR, PO, STI, IIA, and CONS.

Proof. (i) Let $S \in B^n$. Since S is convex and every "Nash product" $\Pi\, x_i^{\omega_i}$ strictly concave if the number of indices i is greater than 1, we have, for every $\ell \in \{1,2,\ldots,k\}$ and all $x,y \in S^\ell$, that $x_i = y_i$ for all $i \in N^1 \cup N^2 \cup \ldots \cup N^\ell$. In particular, x=y for all $x,y \in S^k$.

(ii) The straightforward proof of IR, PO, STI, and IIA of ϕ^H is left to the reader. We show that ϕ^H is consistent. Let $S,T \in B^n$, let $\emptyset \neq M \subset N$ with $\emptyset \neq \bar{M} := N \setminus M$, and suppose that $O_{-M}(\phi^H,S) = O_{-M}(\phi^H,T)$. We have to prove

(28.1) $\phi_i^H(S) = \phi_i^H(T)$ for all $i \in \bar{M}$.

Let $M^1 := \{i \in M : \phi_i^H(S) = 0\}$ and $M^2 := \{i \in M : \phi_i^H(S) > 0\}$. By STI, we may suppose $M^2 = \{i \in M : \phi_i^H(S) = 1\}$. Similarly, we partition M into $M^3 := \{i \in M : \phi_i^H(T) = 0\}$ and $M^4 = M \setminus M^3 = \{i \in M : \phi_i^H(T) = 1\}$. Let further $Z := O^N(O_{-M}(\phi^H,S)) = O^N(O_{-M}(\phi^H,T))$. Suppose Z is degenerate for $\bar{M}$, i.e. there is an $i \in \bar{M}$ with $z_i \leqq 0$ for all $z \in Z$. Then $\phi_i^H(S) = \phi_i^H(T) = 0$ which proves (28.1) for such an i. For the determination of $\phi_j^H(S)$ and $\phi_j^H(T)$ for all other j ($\neq i$), we might, by definition of ϕ^H, restrict attention to $\{x \in S : x_i = 0\}$ and $\{x \in T : x_i = 0\}$ from the start; in other words, it is without loss of generality to suppose that Z is nondegenerate for $\bar{M}$.

Now let $V := \mathrm{com}[\mathrm{conv}(Z \cup \{e^{M^1}\}) + e^{M^2}] = \mathrm{comv}(Z \cup \{e^{M^1}\} + e^{M^2})$ and let $W := \mathrm{com}[\mathrm{conv}(Z \cup \{e^{M^3}\}) + e^{M^4}] = \mathrm{comv}(Z \cup \{e^{M^3}\} + e^{M^4})$. Since Z is supposed to be nondegenerate for $\bar{M}$, we have $V,W \in B^N$. Further, we may in view of STI suppose that $e^{M^1} \in S$ and $e^{M^3} \in T$; and also $e^{M^2} = O^N(\phi^H_{M^2}(S)) \in S$ and $e^{M^4} = O^N(\phi^H_{M^4}(T)) \in T$; so $V \subset S$ and $W \subset T$. Since $\phi^H(S) \in V$ and $\phi^H(T) \in W$, we have by IIA :

(28.2) $\phi^H(V) = \phi^H(S), \phi^H(W) = \phi^H(T)$.

Since we may restrict attention to Z for the determination of both $\phi^H(V)$ and $\phi^H(W)$, we conclude $\phi^H_{\bar{M}}(V) = \phi^H_{\bar{M}}(W)$, which in combination with (28.2) gives (28.1). ∎

Definition 28.9. For $H \in \mathcal{H}^N$, we call $\phi^H : B^n \to \mathbb{R}^n$, defined as in Lemma 28.8 (ii), the *bargaining solution corresponding to the weighted hierarchy* H.

The main purpose of the remainder of this section is to show that the converse of Lemma 28.8 (ii) also holds, that is :

Proposition 28.10. Let $\phi : B^n \to \mathbb{R}^n$ be a bargaining solution satisfying IR, PO, STI, IIA, and CONS. Then there exists a weighted hierarchy $H \in \mathcal{H}^N$ such that $\phi = \phi^H$.

Definition 28.11. We call a bargaining solution $\phi : B^n \to \mathbb{R}^n$ a *Generalized Nash* (GN-)*solution* if it satisfies IR, PO, STI, IIA, and CONS.

We denote $\Delta^M := \mathrm{comv}\{e^i \in \mathbb{R}^M : i \in M\}$, for $\emptyset \neq M \subset N$, and call $\bar{\Delta}^M := \mathrm{com}(E^N(\Delta^M))$ *the standard bargaining game for* $M \subset N$. We shall characterize a GN-solution ϕ by the outcomes it assigns to standard bargaining games.

Definition 28.12. Let $\phi : B^n \to \mathbb{R}^n$ be an n-person bargaining solution and $H = \langle N^1, N^2, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$ a weighted hierarchy. We say that ϕ *determines* H (*on standard bargaining games*) if

$$(28.3) \quad \phi(\Delta^N) = O^N(\omega_{N^1}), \phi(\bar{\Delta}^{N\setminus N^1}) = e^{N^1} + O^N(\omega_{N^2}), \ldots,$$

$$\phi(\bar{\Delta}^{N\setminus(N^1 \cup \ldots \cup N^\ell)}) = e^{N^1\cup\ldots\cup N^\ell} + O^N(\omega_{N^{\ell+1}}), \ldots,$$

$$\phi(\bar{\Delta}^{N\setminus(N^1 \cup \ldots \cup N^{k-1})}) = e^{N^1\cup\ldots\cup N^{k-1}} + O^N(\omega_{N^k}).$$

Lemma 28.13. (i) For every $H \in \mathcal{H}^N$, ϕ^H determines H. (ii) If the bargaining solution $\phi : B^n \to \mathbb{R}^n$ determines H and H' in $\mathcal{H}^N$, then H = H'. (iii) Every Pareto optimal and individually rational bargaining solution $\phi : B^n \to \mathbb{R}^n$ determines some $H \in \mathcal{H}^N$.

Proof. (i) follows with the aid of Lemma 28.14 below. (ii) follows from definition, i.e. (28.3). Similarly, (iii) follows from (28.3), as follows. Let ϕ satisfy IR and PO. Let $N^1 := \{i \in N : \phi_i(\Delta^N) > 0\}$ and, for $i \in N^1$, $\omega_i := \phi_i(\Delta^N)$. If $N \neq N^1$, then let $N^2 := \{i \in N\setminus N^1 : \phi_i(\bar{\Delta}^{N\setminus N^1})) > 0\}$ and, for $i \in N^2$, $\omega_i := \phi_i(\bar{\Delta}^{N\setminus N^1}))$. Etc. ■

The following lemma is the analogue, for the case $m \geq 2$, of Lemma 12.1 :

Lemma 28.14. Let $S \in B^m$, $\omega \in \mathbb{R}^m_{++}$ with $\Sigma^m_{i=1}\omega_i = 1$, $z \in P(S) \cap \mathbb{R}^m_{++}$. Then z maximizes the product $\Pi^m_{i=1} x_i^{\omega_i}$ on $P(S) \cap \mathbb{R}^m_+$ if and only if the hyperplane with equation $\Sigma^m_{i=1} \omega_i z_i^{-1} x_i = 1$ supports S at z.

Proof. Analogous to the proof of Lemma 12.1. ∎

We will prove Proposition 28.10 by induction on the number of players. Therefore, we start by reconsidering the 2-person case.

Lemma 28.15. A 2-person solution $\phi : B \to \mathbb{R}^2$ is a GN-solution if and only if $\phi = \phi^H$ for some $H \in \mathcal{H}^{\{1,2\}}$.

Proof. This lemma is a restatement of Theorem 11.8. For, note that every Pareto optimal solution $\phi : B \to \mathbb{R}^2$ is consistent. Note further that, for $0 < t < 1$, N^t corresponds to the weighted hierarchy $<\{1,2\},(t,1-t)>$, that $N^1 = D^1$ corresponds to $<\{1\},\{2\},(1,1)>$, and that $N^0 = D^2$ corresponds to $<\{2\},\{1\},(1,1)>$. ∎

In the proof of Proposition 28.10 we shall use the following induction hypothesis.

(I) For all $2 \leq k < n$, for all $K \subset N$ such that $|K| = k$, and for all $H \in \mathcal{H}^K$, if the k-person GN-solution $\phi : B^K \to \mathbb{R}^K$ determines H, then $\phi = \phi^H$.

For an n-person bargaining solution $\phi : B^N \to \mathbb{R}^N$, we define solutions for subclasses of the player set N, as follows.

Definition 28.16. Let $\phi : B^N \to \mathbb{R}^N$ be an n-person bargaining solution, and $\emptyset \neq M \subset N$, $m = |M|$. We denote by $M\phi : B^M \to \mathbb{R}^M$ the m-person bargaining solution defined by $M\phi(S) := \phi_M(\mathrm{com}(E^N(S)))$ for every $S \in B^M$.

Lemma 28.17. Let $\phi : B^N \to \mathbb{R}^N$ be a GN-solution, and $\emptyset \neq M \subset N$. Then $M\phi : B^M \to \mathbb{R}^M$ is a GN-solution.

Proof. We only show that $M\phi$ is consistent, and leave verification of the other properties to the reader. Let $S, T \in B^M$, $\emptyset \neq L \subset M$, $L \neq M$, such that $O_{-L}(M\phi,S) = O_{-L}(M\phi,T)$. Then $O_{-L}(\phi,\mathrm{com}(E^N(S))) = O_{-L}(\phi,\mathrm{com}(E^N(T)))$, so by CONS of ϕ we have $M\phi_j(S) = \phi_j(\mathrm{com}(E^N(S))) = \phi_j(\mathrm{com}(E^N(T))) = M\phi_j(T)$ for all

$j \in M$, $j \notin L$. We conclude that $M\phi$ is consistent. ■

Lemma 28.18. Let the bargaining solution $\phi : B^N \to \mathbb{R}^N$ determine $\langle N^1,N^2,\ldots,N^k,\omega\rangle \in \mathcal{H}^N$, where $k \geqq 2$. Let $M := N\backslash N^1$. Then $M\phi$ determines $\langle N^2,N^3,\ldots,N^k,\omega_M\rangle$ $\mathcal{H}^M$.

Proof. $M\phi(\Delta^M) = \phi_M(\bar{\Delta}^M)) = o^M(\omega_{N^2})$,

$M\phi(\mathrm{com}(E^M(\Delta^{M\backslash N^2}))) = \phi_M(\bar{\Delta}^{M\backslash N^2}) = \phi_M(\bar{\Delta}^{N\backslash(N^1\cup N^2)}) =$

$e^{N^2} + o^M(\omega_{N^3})$, etc. ■

The next lemma is Proposition 28.10 for a special case.

Lemma 28.19. Let the GN-solution $\phi : B^N \to \mathbb{R}^N$ determine $H = \langle N,\omega\rangle \in \mathcal{H}^N$. Then $\phi = \phi^H$.

Proof. E.g., theorem 3 in Roth (1979). ■

Whereas the previous lemma treated the special case in which the first class of the weighted hierarchy determined by the GN-solution ϕ contains all players, the following lemma deals with the case in which this first class contains exactly one player.

Lemma 28.20. Let the GN-solution $\phi : B^N \to \mathbb{R}^N$ determine $H = \langle N^1,N^2,\ldots,N^k,\omega\rangle \in \mathcal{H}^N$ with $|N^1| = 1$. Then, under induction hypothesis (I), $\phi = \phi^H$.

Proof. W.l.o.g. let $N^1=\{1\}$. Take $S\in B^N$, and let $z\in P(S), z\geq 0, z_1<h_1(S)$. Take $\beta>0$ so large that $z \in \mathrm{comv}\{h_1(S)e^1, \beta e^i \in \mathbb{R}^N : i=2,3,\ldots,n\} =: V$, which is possible since $z_1 < h_1(S)$. By STI and $\phi(\Delta^N) = e^1$, we have $\phi(V) = h_1(S)e^1$. From this, we conclude $\phi(S) \neq z$ since otherwise, by IIA, $\phi(V \cap S) = h_1(S)e^1$ as well as $\phi(V \cap S) = z$, which would contradict $h_1(S) > z_1$. We have proved :

(28.4) For every $T \in B^N$, $\phi_1(T) = \phi_1^H(T)$ $(= h_1(T))$.

Let $M := N\backslash\{1\}$, and let $L \subset M$ such that $O_{-1}(\phi,S)$ is nondegenerate for L and strongly degenerate for $M\backslash L$. Suppose $M\backslash L \neq \emptyset$. Note that $\phi_i(S) = 0$ for all $i \in M\backslash L$, and that, in view of (28.4), $O_{-\{1\}\cup M\backslash L}(\phi,S) = O_{-\{1\}\cup M\backslash L}(\phi,e^{M\backslash L} + S)$, so CONS gives $\phi_L(S) = \phi_L(e^{M\backslash L} + S)$. In other words, it is without loss of generality to suppose $M\backslash L = \emptyset$, i.e. $L = M$ which means that $O_{-1}(\phi,S)$ is non-degenerate for $M = N\backslash\{1\}$. In view of STI we may suppose $h_1(S) = 1$, so by IIA :

$\phi(S) = \phi(\mathrm{com}(E^N(O_{-1}(\phi,S))))$, hence $\phi_M(S) = M\phi(O_{-1}(\phi,S)) = \phi_M^H(S)$, where the last equality follows from lemmas 28.17 and 28.18, and induction hypothesis (I). So we have proved $\phi_i(S) = \phi_i^H(S)$ for all $i \in N$. ■

We proceed by considering the case, in Proposition 28.10, in which the first class of the weighted hierarchy determined by the GN-solution ϕ contains at least two, but not all players.

Lemma 28.21. Let the GN-solution $\phi : B^N \to \mathbb{R}^N$ determine $H = \langle N^1, N^2, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$, with $1 < |N^1| < n$. Let $S \in B^N$, and $z = \phi(S)$. Then :

(28.5) $\Pi_{i \in N^1} z_i^{\omega_i} = \max\{\Pi_{i\in N^1} x_i^{\omega_i} : x \in S \cap \mathbb{R}_+^N\}$.

Proof. W.l.o.g. let $N^1 = \{1,2,\ldots,s\}$ with $1 < s < n$. Let $M := N\backslash N^1$ and let $q \in S$ with $q_M = 0$ and $\Pi_{i=1}^s q_i^{\omega_i} = \max\{\Pi_{i=1}^s x_i^{\omega_i} : x \in S \cap \mathbb{R}_+^N,\ x_M = 0\}$. As a consequence of Lemma 28.14, there is a hyperplane Y in $\mathbb{R}^{N^1}$, supporting $\{x_{N^1} \in \mathbb{R}^{N^1} : x \in S,\ x_M = 0\}$ at q_{N^1}, with equation $\Sigma_{i=1}^s \omega_i q_i^{-1} x_i = 1$. In view of STI, we may suppose that $q_i = \omega_i (i=1,2,\ldots,s)$. Let $\bar{z} := O^N(z_{N^1})$. We distinguish three cases.

Case (i). $z_{N^1} = q_{N^1}$ $(= \omega_{N^1})$. Then $\Pi_{i=1}^s z_i^{\omega_i} = \Pi_{i=1}^s q_i^{\omega_i} = \max\{\Pi_{i=1}^s x_i^{\omega_i} : x \in S \cap \mathbb{R}_+^N,\ x_M = 0\} = \max\{\Pi_{i=1}^s x_i^{\omega_i} : x \in S \cap \mathbb{R}_+^N\}$.

So for this case (28.5) holds.

Case (ii). $z_{N^1} \notin Y$. Then $z_{N^1} \in \mathrm{int}(\Delta^{N^1})$, so $\bar{z} \in \mathrm{relint}(O^N(\Delta^{N^1}))$. Therefore we can find $\delta > 0$ so large that $z \in V := \mathrm{comv}(O^N(\Delta^{N^1}) \cup \{\delta e^i \in \mathbb{R}^N : i \in M\}) \in B^N$. By STI and the equalities $\phi(\Delta^N) = O^N(\omega_{N^1}) = q$, we have $\phi(V) = q$. Then, by IIA, $\phi(V \cap S) = q$, and also $\phi(V \cap S) = z$. In particular we have $q_{N^1} = z_{N^1}$ and $z_{N^1} \in Y$. From this contradiction we conclude that case (ii) cannot occur.

Case (iii). $z_{N^1} \in Y$, $z_{N^1} \neq q_{N^1}$. In this case, let $y \in S$ with $y_M = 0$ and $y_{N^1} = \frac{1}{2}(z_{N^1} + q_{N^1})$. Then $\Pi_{i=1}^s y_i^{\omega_i} = \max\{\Pi_{i=1}^s x_i^{\omega_i} : x \in a\Delta^{N^1}\}$ where $a \in \mathbb{R}_{++}^{N^1}$ is defined by $a_i = y_i q_i^{-1}$ for every $i=1,2,\ldots,s$. A tedious but elementary calculation then shows that $z_{N^1} \in \mathrm{int}(a\Delta^{N^1})$, so $\bar{z} \in \mathrm{relint}(O^N(a\Delta^{N^1}))$, which brings us in a case analogous to case (ii) above. The conclusion that also case (iii) cannot occur, completes the proof. ■

Lemma 28.22. Let the GN-solution $\phi : B^N \to \mathbb{R}^N$ determine $H = \langle N^1, N^2, \ldots, N^k, \omega\rangle \in \mathcal{H}^N$, with $1 < |N^1| < n$. Then, under (I), $\phi = \phi^H$.

Proof. Let $S \in B^N$. In view of Lemma 28.21, we have :

$$(28.6) \quad \phi_{N^1}(S) = \phi^H_{N^1}(S).$$

By (28.6) and an argument analogous to the one used in the proof of Lemma 28.20, which was based on the consistency of ϕ, we may suppose without loss of generality that $O_{-N^1}(\phi,S)$ is nondegenerate for $M := N\backslash N^1$. In view of STI we may further suppose that $\phi_{N^1}(S) = e^{N^1} \in \mathbb{R}^{N^1}$, so by IIA :

$\phi(S) = \phi(\mathrm{com}(E^N(O_{-N^1}(\phi,S))))$, hence $\phi_M(S) = M\phi(O_{-N^1}(\phi,S)) = \phi^H_M(S)$ where the last equality follows from Lemmas 28.17 and 28.18, and induction hypothesis (I). So we have proved $\phi_i(S) = \phi^H_i(S)$ for all $i \in N$. ■

Proof of Proposition 28.10. Let $H \in \mathcal{H}^N$ be the weighted hierarchy determined by ϕ (cf. Lemma 28.13), say $H = \langle N^1, N^2, \ldots, N^k, \omega\rangle$ with $k \geq 1$. If $k = 1$, then $\phi = \phi^H$ by Lemma 28.19. If $k > 1$ and $|N^1| = 1$, then $\phi = \phi^H$ by Lemmas 28.20 and 28.15. Finally, if $k > 1$ and $1 < |N^1| < n$, then $\phi = \phi^H$ by Lemmas 28.22 and 28.15. ■

Our main result in this section is the following theorem.

Theorem 28.23. The bargaining solution $\phi : B^n \to \mathbb{R}^n$ satisfies IR, PO, STI, IIA, and CONS, if and only if $\phi = \phi^H$ for some $H \in \mathcal{H}^N$.

Proof. Lemma 28.8 (ii) and Proposition 28.10. ■

We conclude with a few remarks.

Remark 28.24. In Peters (1983a), a modification of Theorem 28.23 is proved : degenerate "bargaining games" are allowed there, but on the other hand, the property "degeneracy consistency" which plays the role of CONS, in that paper, seems weaker than CONS. We further conjecture that CONS in Theorem 28.23 may be replaced by some continuity property like PCO.

Remark 28.25. In several papers, W. Thomson characterizes bargaining solutions with the aid of properties which say something about the behavior of n-person bargaining solutions when the number of players n varies. Our consistency property is closely related to his "stability" property : see e.g., Thomson (1982). See also Lensberg (1982), and the following remark.

Remark 28.26. Example 28.3 shows that Theorem 28.23 does not hold without the consistency property. A variation on Theorem 28.23, however, is the following assertion :

(28.7) The bargaining solution $\phi : B^n \to \mathbb{R}^n$ satisfies SIR, PO, STI, and IIA, if and only if $\phi = \phi^H$ for some $H \in \mathcal{H}^N$ of the form $\langle N,\omega\rangle$.

For a proof, see Theorem 3 in Roth (1979); cf. Lemma 28.19. So, if we restrict attention to strongly individually rational solutions, we can dispense with the consistency property. Another question we may ask is : can we dispense with the IIA-property in Theorem 28.23 ? The answer is "no", as the following example of a solution $\phi : B^3 \to \mathbb{R}^3$ shows. For every $S \in B^3$ such that $aS = \text{comv}\{(\frac{3}{2},\frac{3}{2},0),(1,1,1)\}$ for some $a \in \mathbb{R}^3_{++}$, we have $\phi(S) = (a_1^{-1},a_2^{-1},a_3^{-1})(\frac{5}{4},\frac{5}{4},\frac{1}{2})$. For every other game S in B^3, we have $\phi(S) = \phi^H(S)$ with $H = \langle\{1,2,3\},(\frac{1}{3},\frac{1}{3},\frac{1}{3})\rangle \in \mathcal{H}^{\{1,2,3\}}$. Then the reader may verify that this ϕ satisfies PO,SIR, SYM, CONS, but not IIA. This example is taken from Lensberg (1982, Proposition 5.2). Lensberg gives a characterization of the solution $\phi^H : B^n \to \mathbb{R}^n$ with $H = \langle N, (\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})\rangle \in \mathcal{H}^N$, for any $n \in \mathbb{N}$, without using the IIA-property : instead, he uses a stability property which is closely related to CONS, but which is required for a "solution" assigning outcomes to bargaining games in which the number of players may be any arbitrary natural number. See also Remark 28.25.

Remark 28.27. A property closely related to CONS, was already used by Harsanyi (1959). See also Aumann and Maschler (1985).

29. INDIVIDUAL MONOTONICITY

In this section we give an n-person version of Theorem 15.15, in which a family of 2-person individually monotonic bargaining solutions is characterized. We start with the definition of the individual monotonicity property for the n-person case.

Definition 29.1. An n-person bargaining solution $\phi : C^n \to \mathbb{R}^n$, where $C^n \subset B^n$, is called *individually monotonic* if, for all S and T in C^n with $S \subset T$ and every $i \in N$ with $h_j(S) = h_j(T)$ for every $j \in N$, $j \neq i$, we have $\phi_i(S) \leq \phi_i(T)$. (Individual monotonicity : IM)

The following definition is the n-person analogue of Def. 15.11.

Definition 29.2. Let 1_n denote the vector in $\mathbb{R}^n$ with every coordinate equal to 1, and let ∇^n denote the set

$$\{x \in \mathbb{R}^n : 0 \leq x \leq 1_n,\ 1 \leq \Sigma_{i=1}^n x_i \leq n\}.$$

An (*n-person*) *monotonic curve* is a map $\lambda : [1,n] \to \nabla^n$ satisfying the following condition :

(29.1) For all $1 \leq s \leq t \leq n$ we have $\lambda(s) \leq \lambda(t)$ and $\Sigma_{i=1}^n \lambda_i(s) = s$.

By Λ^n we denote the family of all (n-person) monotonic curves. Note that $\Lambda^2 = \Lambda$, cf. Def. 15.11.

Every map $\lambda \in \Lambda^n$ is continuous since $\Sigma_{i=1}^n |\lambda_i(t) - \lambda_i(s)| = |t-s|$ for all $s, t \in [1,n]$. With each map $\lambda \in \Lambda^n$ we would like to associate a bargaining solution, just like we have done in section 15 (Definition 15.12). Unfortunately, difficulties arise if we want that bargaining solution to be Pareto optimal, as the following example will illustrate.

Example 29.3. Let $S := \text{comv}\{(1,0,1),(0,1,1)\} \in B^3$, and let $\lambda_* \in \Lambda^3$ be defined by $\lambda_*(t) = (\frac{t}{3},\frac{t}{3},\frac{t}{3})$ for every $t \in [1,3]$.

Then $\{\lambda_*(t) : t \in [1,3]\} \cap W(S) = \{(\frac{1}{2},\frac{1}{2},\frac{1}{2})\}$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2}) \notin P(S)$.

We will circumvent the difficulty indicated by Example 29.3 by restricting ourselves to a subclass G^n of B^n, to be described below. For other approaches to the same problem, we refer the reader to the end of this section, below Theorem 29.17.

Definition 29.4. G^n denotes the subclass of B^n consisting exactly of those bargaining games S in B^n which satisfy the following condition :

(29.2) For each $x \in S$, $x \geq 0$, and each $i \in \{1,2,\ldots,n\}$ we have : if $x \notin P(S)$ and $x_i < h_i(S)$, then there exists an $\varepsilon > 0$ such that $x + \varepsilon e^i \in S$. (As before, $e^i \in \mathbb{R}^n$ denotes the vector with i-th coordinate 1 and the other coordinates 0.)

In the following lemma, we collect some properties of bargaining games in G^n. There we use for an $S \in G^n$ the notation

$\alpha(x,v) := \sup\{\beta \in [0,\infty) : x + \beta v \in S\}$

where $x \in S$ and $v \in \mathbb{R}^n_+$, $v \neq 0$.

Lemma 29.5. Let $S \in G^n$, $x \in \mathbb{R}^n_+ \cap S$, and $v \in \mathbb{R}^n_+$, $v \neq 0$. Then

(i) $x + \beta v \in S$ for each $\beta \in [0,\alpha(x,v)]$,

(ii) $x + \alpha(x,v)v \in P(S)$ or there is an $i \in \{1,2,\dots,n\}$ with $v_i > 0$ and $x_i + \alpha(x,v)v_i = h_i(S)$,

(iii) $\alpha(0,h(S))h(S) \in P(S)$.

Proof. (i) The set $\{\beta \in [0,\infty) : x + \beta v \in S\}$ is a closed and bounded interval containing 0, since S is compact and convex and $v \neq 0$. Hence, $\alpha(x,v) \in \mathbb{R}$ and $x + \beta v \in S$ for all $\beta \in [0,\alpha(x,v)]$.

(ii) Suppose $x + \alpha(x,v)v \notin P(S)$ and $x_i + \alpha(x,v)v_i < h_i(S)$ for all $i \in N$ with $v_i > 0$. Because $x + \alpha(x,v)v \in S \cap \mathbb{R}^n_+$ (by (i)), there is, in view of (29.2), an $\varepsilon > 0$ such that

$(x + \alpha(x,v)v) + \varepsilon v_i e^i \in S$ for all $i \in N$.

Then $x + (\alpha(x,v) + n^{-1}\varepsilon)v = n^{-1}\Sigma_{i=1}^n (x + \alpha(x,v)v) + \varepsilon v_i e^i$. Hence $x + (\alpha(x,v) + n^{-1}\varepsilon)v \in S$ because S is convex. But that is in contradiction with the definition of $\alpha(x,v)$. So we have proved (ii).

(iii) This follows from (ii) with 0 in the role of x and $h(S)$ in the role of v, if we note that $h(S) > 0$. ∎

The following lemma gives a characterization of the subclass G^n. One reason to present it here is, that it may contribute to the reader's insight; further, it will be used in the proof of Lemma 29.10. Another notation : $\Box^n$ denotes $\text{com}\{1_n\}$, in particular $\Box^2 = \Box$ (see section 21).

Lemma 29.6. For every $S \in B^n$, we have $S \in G^n$ if and only if $S = h(S)\Box^n$ or $S \cap \text{cl}(h(S)\Box^n \setminus S) \cap \mathbb{R}^n_+ = P(S) \cap \mathbb{R}^n_+$.

Proof. Let $S \in B^n$. W.l.o.g. we assume $h(S) = 1_n$. We first show the "only if"-part of the lemma. Let $S \in G^n$, and suppose $S \neq \Box^n$. Let $x \in P(S)$, $x \geqq 0$. Then $y \in \Box^n \setminus S$ for every $y \in \text{conv}\{x,1_n\}$ with $y \neq x$, so $x \in \text{cl}(\Box^n \setminus S)$. We conclude that $P(S) \cap \mathbb{R}^n_+ \subset S \cap \text{cl}(\Box^n \setminus S) \cap \mathbb{R}^n_+$.

Next, let $x \in S\backslash P(S)$, $x \geqq 0$. By (29.2), there exists for every $i \in N$ with $x_i < 1$, an $\varepsilon^i > 0$ such that $x + \varepsilon^i e^i \in S$. Let $\varepsilon := \min\{\varepsilon^i : i \in N \text{ with } x_i < 1\}$ and let Q be the ball with center x and radius $n^{-1}\varepsilon$. Then, for $y \in Q \cap \square^n$, we have $y_i \leqq x_i$ if $x_i = 1$ and $y_i \leqq x_i + n^{-1}\varepsilon$ if $x_i < 1$. Let, for $i \in N$, $x^i \in S$ be defined by $x^i := x$ if $x_i = 1$ and $x^i := x + \varepsilon^i e^i$ if $x_i < 1$. Because S is convex, $n^{-1}\sum_{i=1}^n x^i \in S$. Now $y \in Q \cap \square^n$ implies $y \leq n^{-1}\sum_{i=1}^n x^i$, so $y \in S$. We have shown that $Q \cap \square^n \subset S$, hence $Q \cap \square^n = Q \cap S$, which implies $x \notin cl(\square^n\backslash S)$. We have proved : $S \cap cl(\square^n\backslash S) \cap \mathbb{R}^n_+ \subset P(S) \cap \mathbb{R}^n_+$, which completes the proof of the "only if"-part.

For the "if"-part : if $S = \square^n$, then $S \in G^n$ straightforwardly; now suppose $S \neq \square^n$, and $S \cap cl(\square^n\backslash S) \cap \mathbb{R}^n_+ = P(S) \cap \mathbb{R}^n_+$. Let $x \in S$, $x \geqq 0$, $x \notin P(S)$, and $i \in N$ with $x_i < 1 = h_i(S)$. Then $x \notin cl(\square^n\backslash S)$. For every $\varepsilon \in [0,1-x_i]$, we have $x + \varepsilon e^i \in \square^n$; so $x + \varepsilon e^i \notin S$ for every $\varepsilon \in (0,1-x_i]$ would imply $x \in cl(\square^n\backslash S)$, and hence a contradiction. We have proved (29.2), so $S \in G^n$. ■

Our purpose is, to describe all n-person bargaining solutions $\phi : G^n \to \mathbb{R}^n$ which satisfy - on G^n - the properties PO, STI, and IM. Just like in section 15, IM and PO imply IR :

<u>Lemma 29.7.</u> Let the bargaining solution $\phi : G^n \to \mathbb{R}^n$ satisfy IM and PO. Then ϕ satisfies IR.

<u>Proof.</u> Analogous to the proof of Lemma 15.7. ■

Again, just as in section 15, the IM-property is strongly related to the following property.

<u>Definition 29.8.</u> An n-person bargaining solution $\phi : G^n \to \mathbb{R}^n$ is called *restrictedly monotonic* if, for all S and T in G^n with $S \subset T$ and $h(S) = h(T)$, we have $\phi(S) \leqq \phi(T)$.

(Restricted monotonicity : RM)

<u>Lemma 29.9.</u> Let $\phi : G^n \to \mathbb{R}^n$ be a bargaining solution satisfying PO and STI. Then ϕ satisfies IM if and only if ϕ satisfies RM.

<u>Proof.</u> Analogous to the proof of Lemma 15.10. ■

We shall associate an n-person bargaining solution with each monotonic curve $\lambda \in \Lambda^n$. In order to do this, we need the following lemma.

Lemma 29.10. For each $\lambda \in \Lambda^n$ and $S \in G^n$ with $h(S) = 1_n$, the set $P(S) \cap \{\lambda(t) : t \in [1,n]\}$ contains exactly one point.

Proof. Let $\lambda \in \Lambda^n$ and $S \in G^n$ with $h(S) = 1_n$, and denote $L := \{\lambda(t) : t \in [1,n]\}$. In view of condition (29.1) and the definition of $P(S)$, the set $L \cap P(S)$ contains at most one point.
Let $m := \sup\{t \in [1,n] : \lambda(t) \in S\}$. From $\lambda(1) \in S$, the continuity of λ and the closedness of S, we conclude $\lambda(m) \in S$. If $m=n$, then $L \cap P(S) = \{1_n\}$, and the proof is completed. Otherwise, $\{\lambda(t) : t \in (m,n]\} \subset \square^n \setminus S$, hence $\lambda(m) \in cl(\square^n \setminus S)$. So by Lemma 29.6 we conclude that $\lambda(m) \in P(S)$. ∎

Definition 29.11. For each $\lambda \in \Lambda^n$, we denote by π^λ the n-person bargaining solution : $G^n \to \mathbb{R}^n$ which assigns to every $S \in G^n$ with $h(S) = 1_n$ the unique point in $P(S) \cap \{\lambda(t) : t \in [1,n]\}$ (cf. Lemma 29.10), and to every other $S \in G^n$ the point $h(S)z$ with $z = \pi^\lambda((h_1(S)^{-1}, h_2(S)^{-1}, \ldots, h_n(S)^{-1})S)$. We call π^λ the *bargaining solution corresponding to* $\lambda \in \Lambda^n$.

Verification of the following proposition is left to the reader.

Proposition 29.12. For every $\lambda \in \Lambda^n$, the n-person bargaining solution $\pi^\lambda : G^n \to \mathbb{R}^n$ satisfies PO, STI, and RM.

The converse of Proposition 29.12 is also true :

Proposition 29.13. Let the bargaining solution $\phi : G^n \to \mathbb{R}^n$ satisfy PO, STI, and RM. Then $\phi = \pi^\lambda$ for some $\lambda \in \Lambda^n$.

Proof. For each $t \in [1,n]$ let

$$V(t) := \{x \in \mathbb{R}^n : x \leqq 1_n, \Sigma_{i=1}^n x_i \leqq t\}.$$

Then $V(t) \in G^n$ and $h(V(t)) = 1_n$ for each $t \in [1,n]$.
Define $\lambda : [1,n] \to \mathbb{R}^n$ by $\lambda(t) := \phi(V(t))$ for each $t \in [1,n]$. For $1 \leqq s \leqq t \leqq n$ we have by RM : $\lambda(s) = \phi(V(s)) \leqq \phi(V(t)) = \lambda(t)$. Furthermore, for each $t \in [1,n]$, $\lambda(t) \in P(V(t)) = \{x \in V(t) : \Sigma_{i=1}^n x_i = t\}$. Hence, $\Sigma_{i=1}^n \lambda_i(t) = t$. So $\lambda \in \Lambda^n$. Note that

(29.3) $\phi(V(t)) = \pi^\lambda(V(t))$ for each $t \in [1,n]$.

We want to prove that $\phi = \pi^{\lambda}$. In view of STI it is sufficient to show that $\phi(S) = \pi^{\lambda}(S)$ where $S \in G^n$ with $h(S) = 1_n$. Let $s := \Sigma_{i=1}^{n} \pi_i^{\lambda}(S)$, and let $W := V(s) \cap S$. Then $W \in G^n$ with $h(W) = 1_n$.
Since $\pi^{\lambda}(S) \in P(S) \cap P(V(s))$, we have in view of (29.3) :

(29.4) $\quad \pi^{\lambda}(W) = \pi^{\lambda}(S) = \pi^{\lambda}(V(s)) = \phi(V(s)) \in P(W) \cap P(S) \cap P(V(s))$.

In view of RM, $\phi(W) \leq \phi(V(s))$. Since, by (29.4), $\phi(V(s)) \in P(W)$, we obtain

(29.5) $\quad \phi(W) = \phi(V(s))$.

In view of RM, $\phi(W) \leq \phi(S)$. By (29.4) and (29.5) : $\phi(W) \in P(S)$. So

(29.6) $\quad \phi(W) = \phi(S)$.

Combining (29.4) - (29.6) we may conclude that $\phi(S) = \pi^{\lambda}(S)$. ■

The main result of this section is the following theorem.

Theorem 29.14. The n-person bargaining solution $\phi : G^n \to \mathbb{R}^n$ satisfies PO, STI, and IM if and only if $\phi = \pi^{\lambda}$ for some $\lambda \in \Lambda^n$.
Proof. Lemma 29.9 and Propositions 29.12 and 29.13. ■

Remark 29.15. Let $\lambda_* \in \Lambda^n$ be defined by $\lambda_*(t) := tn^{-1}1_n$ for every $t \in [1,n]$. Then the solution π^{λ_*} corresponding to λ_* can been seen as an extension, on G^n, of the (2-person) Kalai - Smorodinsky solution KS (see Def. 15.3). It is easily seen that π^{λ_*} is the only symmetric member of $\{\pi^{\lambda} : \lambda \in \Lambda^n\}$.

Remark 29.16. Theorem 15.15, which characterizes the family $\{\pi^{\lambda} : \lambda \in \Lambda = \Lambda^2\}$, is a special case of Theorem 29.14 : for, if n=2, then $G^n = B^n$. This follows simply by verification of (29.2) for B^2, or by Lemma 29.6.

Just like in the 2-person case (cf. Corollary 15.16) we can ask the question : Which solutions $\phi : G^n \to \mathbb{R}^n$ satisfy, besides PO and STI, the properties IM as well as IIA ?

Theorem 29.17. The solution $\phi : G^n \to \mathbb{R}^n$ satisfies PO, STI, IIA, and IM, if and only if there is a permutation $\pi : N \to N$ such that, for every $S \in G^n$, $\pi\phi(S)$ is the lexicographical maximum of $S \cap \mathbb{R}^2_+$.
(Recall the notation introduced before Def. 10.7.)
Proof. For the "if"-part : let π be a permutation of N with, for $S \in G^n$, $\phi(S) \in S$ such that $\pi\phi(S)$ is the lexicographical maximum of $S \cap \mathbb{R}^2_+$. Then

note that $\phi = \phi^H : G^n \to \mathbb{R}^n$ with $H = \langle\{\pi(1)\},\{\pi(2)\},\ldots,\{\pi(n)\},1_n\rangle$ (see section 28). So ϕ satisfies PO, STI, and IIA (Theorem 28.23). Note further that $\phi = \pi^{\lambda_0}$ where $\lambda_0 \in \Lambda^n$ has graph
$\mathrm{conv}\{e^{\pi(1)}, e^{\pi(1)} + e^{\pi(2)}\} \cup \mathrm{conv}\{e^{\pi(1)} + e^{\pi(2)}, e^{\pi(1)} + e^{\pi(2)} + e^{\pi(3)}\} \cup \ldots$
$\cup\ \mathrm{conv}\{\Sigma_{i=1}^{n-1} e^{\pi(i)}, 1_n\}$. So ϕ satisfies IM (Theorem 29.14). For the "only if"-part, let $\lambda \in \Lambda^n$ be such that λ is not of the form λ_0 above. In view of Theorem 29.14, it is sufficient to show that the corresponding $\pi^\lambda : G^n \to \mathbb{R}^n$ does not satisfy IIA. Note that λ is not of the form λ_0 if and only if there is a point on the graph of λ with at least two coordinates unequal to 0 and 1. W.l.o.g. we may suppose, for some $t_0 \in [1,n)$:

(29.7) $\quad 0 < \lambda_1(t_0) < 1,\ 0 < \lambda_2(t_0) < 1.$

We may further suppose that $\lambda_1(t) > \lambda_1(t_0)$ or $\lambda_2(t) > \lambda_2(t_0)$ for all $t > t_0$, say :

(29.8) $\quad \lambda_2(t) > \lambda_2(t_0)$ for all $t > t_0$.

Let $V := \{x \in \square^n : \Sigma_{i=1}^n x_i \leq t_0\}$. Then $V \in G^n$ and $\pi^\lambda(V) = \lambda(t_0)$.
Choose $\alpha \in (\lambda_1(t_0),1)$, and let $W := \{x \in V : x_1 \leq \alpha\}$, then $W \in G^n$.
Suppose, contrary to what we want to prove, that π^λ satisfies IIA. Then, since $W \subset V$ and $\pi^\lambda(V) = \lambda(t_0) \in W$, we have

(29.9) $\quad \pi^\lambda(W) = \lambda(t_0)$.

On the other hand, $\pi^\lambda(W) = (\alpha,1,\ldots,1)\pi^\lambda((\alpha^{-1},1,\ldots,1)W) =$
$= (\alpha,1,\ldots,1)\lambda(t)$ for some $t > t_0$ since $\lambda(t_0) \in ((\alpha^{-1},1,\ldots,1)W)\setminus P((\alpha^{-1},1,\ldots,1)W)$.
So, in view of (29.8) and (29.9), we have $\pi_2^\lambda(W) = \lambda_2(t) > \lambda_2(t_0) = \pi_2^\lambda(W)$, an impossibility. We conclude that π^λ does not satisfy IIA. ■

We present a brief discussion of related literature. The results in this section were published in Peters and Tijs (1983a). Other approaches to the more restricted problem of extending the Kalai - Smorodinsky solution to the case $n > 2$, were taken by Thomson (1980) and Imai (1983). Thomson also restricts the class of bargaining games, by requiring a bargaining game to have no weakly Pareto optimal points in the positive orthant, i.e. for $S \in B^n$ he requires $W(S) \cap \mathbb{R}_+^n \subset P(S)$; this restriction is stronger than our restriction to G^n. Imai, by means of modifying the relevant properties, obtains a characterization of the so-called "lexicographic maxmin solution"; on our class G^n this solution coincides with the symmetric solution $\pi^{\lambda *}$ of Remark 29.15.

We indicated in the beginning of this section (Example 29.3) that an extension of Theorem 15.15 - a characterization of a family of 2-person individually monotonic bargaining solutions - to the n-person case is unlikely to be trivial. We conclude this section with an "impossibility result" which is inspired by Roth (1979, p.105), but stronger than his result because we drop the symmetry property and consider the - smaller - class B^n.

<u>Theorem 29.18.</u> There exists no bargaining solution $\phi : B^n \to \mathbb{R}^n$ $(n>2)$ which is Pareto optimal and individually monotonic.

<u>Proof.</u> For every $i \in N$, let y^i be the vector in $\mathbb{R}^n$ with i-th coordinate 0 and all other coordinates equal to 1, and let

$V^i := \text{comv}\{y^j : j \in N, j \neq i\} \in B^n$. Let further $V := \text{comv}\{y^i : i \in N\}$. Suppose $\phi : B^n \to \mathbb{R}^n$ satisfies PO and IM. By PO, we have $\phi_i(V^i) = 1$ for every $i \in N$. By IM, $\phi_i(V) \geq \phi_i(V^i)$ for every $i \in N$. Hence $\phi(V) \geq 1_n$, an impossibility since $1_n \notin V$. So such a ϕ cannot exist. ∎

30. RISK PROPERTIES

In chapter 8 (sections 23 - 27) we have considered risk properties for 2-person bargaining solutions. We shall now present a few generalizations to the n-person case.

We start with some notations and definitions. Most of these are straightforward extensions of the corresponding notations and definitions for the 2-person case, but we list them anyhow, for completeness'sake.

Let $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n\rangle \in BS^n$ be an n-person bargaining situation. For every $i \in N$, we denote by $C^i(\Gamma)$ the family of all continuous nondecreasing concave functions $k : u^i(A) \to \mathbb{R}$ with $k(0) = 0$ and $k(\lambda) > 0$ for some $\lambda \in u^i(A)$; and, for $k^i \in C^i(\Gamma)$, the bargaining situation

$\langle A,\bar{a},u^1,u^2,\ldots,u^{i-1},k^i \circ u^i,u^{i+1},\ldots,u^n\rangle \in BS^n$ is denoted $K^i(\Gamma)$. We say that $K^i(\Gamma)$ arises from Γ by the replacement of player i by a more risk averse player.

<u>Definition 30.1.</u> Let $C^n \subset B^n$ and $\tilde{C} \subset CS^n$ (cf. notation 10.2). An n-person bargaining solution $\phi : C^n \to \mathbb{R}^n$ is called *risk sensitive on* $\tilde{C}$ if, for all $i, j \in N$ with $i \neq j$, $\Gamma \in \tilde{C}$, $k^i \in C^i(\Gamma)$ with $K^i(\Gamma) \in CS^n$, we have

$\phi_j(S_{K^i(\Gamma)}) \geq \phi_j(S_\Gamma)$. If $\tilde{C} = BS^n$ (and $C^n = B^n$), then ϕ is called *risk sensitive*.

(Risk sensitivity : RS)

Definition 30.2. Let C^n and $\tilde{C}$ be as in Def. 30.1. An n-person bargaining solution $\phi : C^n \to \mathbb{R}^n$ has the *worse alternative property on* $\tilde{C}$, if, for each $i \in N$, $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n \rangle \in \tilde{C}$, $k^i \in C^i(I)$ with $K^i(\Gamma) \in CS^n$, $\ell \in alt(\phi,\Gamma)$, $m \in alt(\phi,K^i(\Gamma))$, we have $Eu^i(\ell) \geq Eu^i(m)$. If $\tilde{C} = BS^n$ (and $C^n = B^n$), ϕ is said to have the *worse alternative property*.

(Worse alternative : WA)

We shall denote by BSC^n the family of n-person bargaining situations such that all Pareto optimal outcomes in the corresponding bargaining games represent riskless alternatives, i.e.

$$BSC^n := \{\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n \rangle \in BS^n : \text{for every } x \in P(S_\Gamma), \text{ there exists an } a \in A \text{ with } x = (u^1(a),u^2(a),\ldots,u^n(a))\}.$$

Note that $BSC^2 = BSC$ (see section 23).

We now give two results, the first of which is the n-person analogue of Lemma 23.5 : it says that bargaining situations Γ in BSC^n "behave nicely" under transformations k^i in $C^i(\Gamma)$. The second result is the n-person analogue of the "impossibiblity" Theorem 23.6, and says that no weakly Pareto optimal and individually rational solution is risk sensitive or has the worse alternative property (that is, on BS^n). Both results can be proved in the same way as the corresponding results in section 23.

Lemma 30.3. Let $\Gamma = \langle A,\bar{a},u^1,u^2,\ldots,u^n \rangle \in BSC^n$, $i \in N$, $k^i \in C^i(\Gamma)$. For $x \in P(S_\Gamma)$ let $\hat{x} \in \mathbb{R}^n$ denote the point with i-th coordinate $k^i(x_i)$ and j-th coordinate x_j (for every $j \neq i$), and let $\hat{P} := \{\hat{x} : x \in P(S_\Gamma)\}$. Then $S_{K^i(\Gamma)} = com(\hat{P})$ and $P(S_{K^i(\Gamma)}) \subset \hat{P}$.

Theorem 30.4. Let $\phi : B^n \to \mathbb{R}^n$ be a weakly Pareto optimal and individually rational n-person bargaining solution. Then ϕ is not risk sensitive, and does not have the worse alternative property.

Also the proof of the next lemma is omitted : cf. Lemma 23.7.

Lemma 30.5. Let $\phi : B^n \to \mathbb{R}^n$ be a Pareto optimal n-person bargaining solution, and $\tilde{C} \subset BS^n$. Then, if ϕ satisfies RS on $\tilde{C}$, ϕ also satisfies WA on $\tilde{C}$.

Lemma 30.5 says that, on any subclass of BS^n, RS implies WA; Theorem 30.4, however, destroys our hope of finding "reasonable" bargaining solutions which have the worse alternative property (on BS^n). Consequently, we shall in most cases restrict attention to (bargaining solutions on) BSC^n, just like we did in chapter 8.

The following theorem describes a property which, for Pareto optimal bargaining solutions on BSC^n, is equivalent to the worse alternative property.

Theorem 30.6. Let $\phi : B^n \to \mathbb{R}^n$ be a Pareto optimal n-person bargaining solution. Then the following two statements are equivalent :

(i) ϕ has the worse alternative property on BSC^n.

(ii) For every $\Gamma \in BSC^n$, $i \in N$, $k^i \in C^i(\Gamma)$, we have

$O_{-i}(\phi,S_\Gamma) \subset O_{-i}(\phi,S_{K^i(\Gamma)})$.

(For this notation, see section 28, before Def. 28.4.)

Proof. Suppose ϕ satisfies WA. Let $\Gamma = \langle A,\bar{a},u^1,u^2,\dots,u^n\rangle \in BSC^n$, $i \in N$, $k^i \in C^i(\Gamma)$, and take $a,b \in A$ with $a \in alt(\phi,\Gamma)$ and $b \in alt(\phi,K^i(\Gamma))$. This is possible because $\Gamma \in BSC^n$, ϕ is Pareto optimal, and in view of Lemma 30.3. Then, by WA, $u^i(a) \geq u^i(b)$. Since both $(u^1(a),u^2(a),\dots,u^n(a))$ and $(u^1(b),u^2(b),\dots,u^n(b))$ are in $P(S_\Gamma)$ by PO of ϕ and Lemma 30.3, this inequality implies $O_{-i}(\phi,S_\Gamma) = \{x_{N\setminus\{i\}} : x \in S_\Gamma, x_i = u^i(a)\} \subset \{x_{N\setminus\{i\}} : x \in S_\Gamma, x_i = u^i(b)\} = O_{-i}(\phi,S_{K^i(\Gamma)})$. The last equality follows from Lemma 30.3. We have proved (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) follows by reversing the argument. ■

Inspired by the equivalence in Theorem 30.6, we introduce the following property.

Definition 30.7. Let C^n and $\tilde{C}$ be as in Definition 30.1. An n-person bargaining solution $\phi : C^n \to \mathbb{R}^n$ has the *risk profit opportunity property on* $\tilde{C}$, if, for each $\Gamma \in \tilde{C}$, $i \in N$, $k^i \in C^i(\Gamma)$ with $K^i(\Gamma) \in CS^n$, we have $O_{-i}(\phi,S_\Gamma) \subset O_{-i}(\phi,S_{K^i(\Gamma)})$. If $\tilde{C} = BS^n$ (and $C^n = B^n$), ϕ is said to have the *risk profit opportunity property.*

(Risk profit opportunity : RPO)

Thus, if a solution has the risk profit opportunity property, then the set of utility (n-1)-tuples in an n-person bargaining game S, available for the players $j \neq i$ if player i receives $\phi_i(S)$, does not decrease if player i is replaced by a more risk averse player. A further explanation of the expression "risk profit opportunity" is given by the observation that (with notations as in Def. 30.7), if ϕ is Pareto optimal, then $O_{-i}(\phi,S_\Gamma) \subset O_{-i}(\phi,S_{K^i(\Gamma)})$ implies either $\phi_j(S_\Gamma) = \phi_j(S_{K^i(\Gamma)})$ for all $j \neq i$, or $\phi_j(S_\Gamma) < \phi_j(S_{K^i(\Gamma)})$ for at least one $j \neq i$. This observation also indicates that RPO can be seen as an extension of the 2-person risk sensitivity property, different from (and weaker than) n-person risk sensitivity. Summarizing some relations between several risk properties for Pareto optimal solutions, we have : on BSC^2 = BSC, WA, RS, and RPO are equivalent (Theorems 30.6, 23.12); on BSC^n, WA and RPO are equivalent and implied by RS (Lemma 30.5 and Theorem 30.6), and also, RPO is properly weaker than RS if $n > 2$, as will appear in the sequel when we study IIA-solutions.

The next theorem corresponds to Theorem 23.13.

<u>Theorem 30.8.</u> Let $\phi : B^n \to \mathbb{R}^n$ be a Pareto optimal and consistent n-person bargaining solution which has the risk profit opportunity property on BSC^n. Then ϕ is scale transformation invariant.

<u>Proof.</u> Let $S \in B^n$, $k \in \mathbb{R}^n_{++}$. We have to prove :

(30.1) $\phi(kS) = k\phi(S)$.

Let Γ be the trivial bargaining situation corresponding to S (see Example 9.6). Then $\Gamma \in BSC^n$. Denote by $k^i \in C^i(\Gamma)$ multiplication by k^i. Since also the inverse function of k^i, i.e. dividing by k^i, is in $C^i(\Gamma)$, we have by double application of RPO :

(30.2) $O_{-i}(\phi,S_\Gamma) = O_{-i}(\phi,S_{K^i(\Gamma)})$ for every $i \in N$.

Now (30.2), the linearity of k^i, and PO, imply $[k_i\phi_i(S_\Gamma)=]k^i(\phi_i(S_\Gamma)) = \phi_i(S_{K^i(\Gamma)})$ for every $i \in N$, hence

(30.3) $k_i\phi_i(S_\Gamma) = \phi_i((1,1,\dots,1,k_i,1,\dots,1)S_\Gamma)$ for every $i \in N$.

From (30,2), (30.3), and CONS of ϕ, we obtain

$k_1\phi_1(S_\Gamma) = \phi_1((k_1,1,\dots,1)S_\Gamma)$ and $\phi_j(S_\Gamma) = \phi_j((k_1,1,\dots,1)S_\Gamma)$ for all $j \neq 1$.

Repeating the whole argument (n-1) more times, we find $k\phi(S_\Gamma) = \phi(kS_\Gamma)$, proving (30.1). ∎

Remark 30.9. By a small modification of the proof of Theorem 30.8, one can show that in that theorem, CONS and RPO may be replaced by one property : RS.

Remark 30.10. We give an example of a solution which satisfies RPO on BSC^3 (and PO), but neither CONS nor STI nor RS on BSC^3. We leave the verification of these claims to the reader. (He may use, among other things, Theorem 24.1.) That solution $\phi : B^3 \to \mathbb{R}^3$ is defined by $\phi_1(S) := \max\{x_1 : x \in S \cap \mathbb{R}^3_+\}$, and $(\phi_2(S),\phi_3(S))$ maximizes $x_2^t x_3^{1-t}$ on $\{x \in S \cap \mathbb{R}^3_+ : x_1 = \phi_1(S)\}$ where $0 < t < 1$ is such that $t(1-t)^{-1} = \phi_1(S)^2$.

Now that we have established some elementary relations between risk properties and properties like Pareto optimality and scale transformation invariance, we proceed by considering risk properties of n-person IIA-solutions (cf. section 28). Our first main result is the following theorem, which is an extension of Theorem 24.1, in view of the equivalence of RPO and RS on BSC^2 (Theorems 30.6 and 23.12).

Theorem 30.11. Let $\phi : B^n \to \mathbb{R}^n$ be a bargaining solution satisfying IR, PO, STI, and IIA. Then ϕ satisfies RPO on BSC^n.

Proof. Let $\Gamma \in BS^n$, $i \in N$, $k^i \in C^i(\Gamma)$. Let $z := \phi(S_\Gamma)$, $\hat{z} := \phi(S_{K^i(\Gamma)})$. We want to prove :

(30.4) $\quad O_{-i}(\phi,S_\Gamma) \subset O_{-i}(\phi,S_{K^i(\Gamma)})$.

First suppose that $\hat{z}_i = 0$. Then $O_{-i}(\phi,S_{K^i(\Gamma)}) = \{x_{N\setminus\{i\}} \in \mathbb{R}^{N\setminus\{i\}} : x \in S_\Gamma$ with $x_i = 0\}$ in view of Lemma 30.3, from which (30.4) follows immediately. Next suppose $\hat{z}_i > 0$. In view of Lemma 30.3 there is a (unique) point y in $P(S_\Gamma)$ with $y_j = \hat{z}_j$ for every $j \neq i$. Since ϕ satisfies STI, we may suppose

(30.5) $\quad \hat{z}_i = k^i(y_i) = y_i, \hat{z} = y$.

The concavity of k^i, (30.5), and $k^i(0) = 0$, then imply :

(30.6) For all λ for which k^i is defined, we have

$$k^i(\lambda) \geqq \lambda \text{ if } \lambda \in [0,y_i],\ k^i(\lambda) \leqq \lambda \text{ if } \lambda \geqq y_i.$$

Now suppose that $\hat{z}_i > z_i$ $(\geqq 0)$ and let

$$T := \text{comv}\{0,z,\hat{z},\alpha 1_n\}$$

where $\alpha \in \mathbb{R}_{++}$ is so small that $\alpha 1_n \in S_\Gamma \cap S_{K^i(\Gamma)}$. Then $T \in B^n$, $T \subset S_\Gamma$ since $\hat{z} \in S_\Gamma$ by (30.5), $T \subset S_{K^i(\Gamma)}$ since $z \in S_{K^i(\Gamma)}$ by (30.6). So by IIA,

$\phi(T) = z = \hat{z}$, contradicting our assumption $\hat{z}_1 > z_1$. Hence, $\hat{z}_1 \leqq z_1$, and then (30.4) follows from (30.5) and Lemma 30.3. ■

A consequence of Theorem 30.11 and Lemma 28.8 (ii) is, that for every weighted hierarchy $H \in \mathcal{H}^N$, the corresponding solution ϕ^H has the risk profit opportunity property on BSC^n. We now present two examples of solutions ϕ^H which are not risk sensitive on BSC^n, for $n=3$.

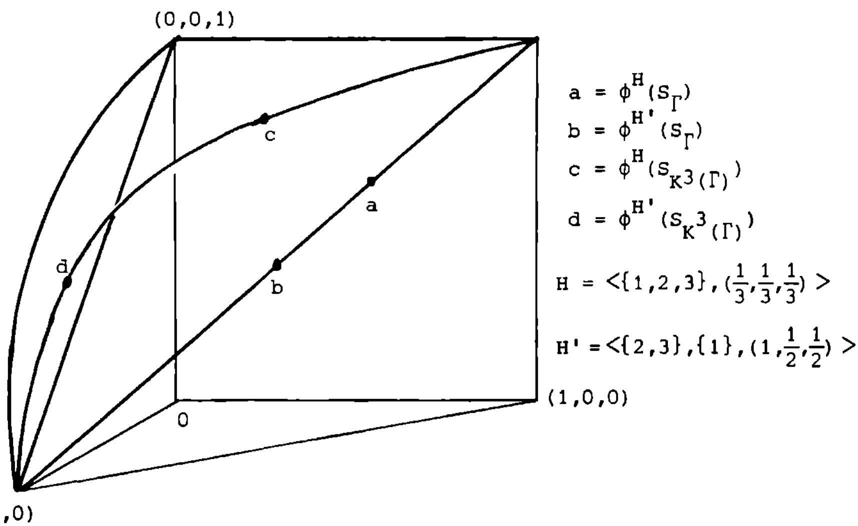
<u>Example 30.12.</u> (See Fig. 30.1.) Let $\Gamma \in BSC^3$ be such that $S_\Gamma = S$ where

$S := \text{comv}\{(1,0,0),(0,1,0),(0,0,1),(1,0,1)\}$.

Then $P(S) = \text{conv}\{(0,1,0),(1,0,1)\}$. Let $k^3 \in C^3(I)$ be defined by $k^3(\lambda) = \sqrt{\lambda}$ for all $\lambda \in [0,1]$. Then $P(S_{K^3(\Gamma)}) = \{(\alpha,1-\alpha,\sqrt{\alpha}) : \alpha \in [0,1]\}$.

(i) Let $H = \langle\{1,2,3\},\omega\rangle \in \mathcal{H}^{\{1,2,3\}}$. Straightforward calculations show : $\omega_1 + \omega_3 = \phi_1^H(S_\Gamma) > \phi_1^H(S_{K^3(\Gamma)}) = (2\omega_1 + \omega_3)(2-\omega_3)^{-1}$, so ϕ^H is not risk sensitive on BSC^3.

(ii) Let $H' = \langle\{2,3\},\{1\},\omega\rangle \in \mathcal{H}^{\{1,2,3\}}$, so $\omega > 0$, $\omega_2 + \omega_3 = 1$, $\omega_1 = 1$. Again, straightforward calculations show : $\omega_3 = \phi_1^{H'}(S_\Gamma) > \phi_1^{H'}(S_{K^3(\Gamma)}) = \omega_3(1 + \omega_2)^{-1}$, so also $\phi^{H'}$ is not risk sensitive on BSC^3.



<u>Figure 30.1.</u>

We use Example 30.12 in the proof of the following theorem.

Theorem 30.13. Let $H = \langle \ldots,\omega\rangle \in \mathcal{H}^N$. The following two statements are equivalent.

(i) ϕ^H is risk sensitive on BSC^n.

(ii) There is a permutation π of N such that either

$H = \langle\{\pi(1)\},\{\pi(2)\},\ldots,\{\pi(n)\},1_n\rangle$

or

$H = \langle\{\pi(1)\},\{\pi(2)\},\ldots,\{\pi(n-2)\},\{\pi(n-1),\pi(n)\},\omega\rangle$.

Proof. We first show the implication (ii) $\Rightarrow$ (i). Let $\Gamma \in BSC^n$. Let π be a permutation of N. If $H = \langle\{\pi(1)\},\{\pi(2)\},\ldots,\{\pi(n)\},1_n\rangle$, then, if a player (say) $\pi(i)$ for $i \in N$, is replaced by a more risk averse player, by definition of ϕ^H the solution outcome changes only (possibly) for player $\pi(i)$. So ϕ^H is risk sensitive on BSC^n. If $H = \langle\{\pi(1)\},\ldots,\{\pi(n-2)\},\{\pi(n-1),\pi(n)\},\omega\rangle$, then, if a player $\pi(i)$ with $i \leq n-2$ is replaced by a more risk averse player then the solution outcome assigned by ϕ^H changes only (possibly) for player $\pi(i)$. If player $\pi(n-1)[\pi(n)]$ is replaced by a more risk averse player, then the solution outcome does not change for all players $\pi(j)$ with $j \leq n-2$, and changes for player $\pi(n)[\pi(n-1)]$ only to that player's advantage (cf. Theorem 24.1, or Theorem 30.11 for n=2 since then RS and RPO are equivalent). So also this ϕ^H is risk sensitive on BSC^n.

Next , we prove the implication (i) $\Rightarrow$ (ii). Suppose that H is not as in (ii). Let $H = \langle N^1,N^2,\ldots,N^\ell,\omega\rangle$. The assumption that H is not as in (ii) implies that either

Case 1 : Some class N^h ($1 \leq h \leq \ell$) contains at least three players

or

Case 2 : Each class contains at most two players, and some class N^h ($1 \leq h < \ell$) contains exactly two players.

Firstly, suppose that H is as in case 1. We may suppose w.l.o.g. that 1,2, and 3 are in N^h. Let S, Γ and k^3 be as in Example 30.12, and let $T := com(E^N(S))$. Then $P(T) = \{E^N(z) : z \in P(S)\}$. The same calculation as in Example 30.12 (i) shows that $\phi_1^H(T) > \phi_1^H(com(E^N(S_{K^3(\Gamma)})))$, which can be seen to imply that ϕ^H is not risk sensitive on BSC^n, e.g. by looking at underlying trivial bargaining situations. Similarly, Example 30.12 (ii) can be used to show that also in case 2, ϕ^H is not risk sensitive on BSC^n. ∎

Theorem 30.13 shows that only a relatively small subclass of $\{\phi^H : H \in H^N\}$ consists of solutions which are risk sensitive on BSC^n. This subclass consists of n! "dictatorial" solutions and a family of "almost-dictatorial" solutions each one determined by a number in (0,1) and an ordered partition of N out of $\frac{1}{2}(n!)$ possible ones.

Next, we consider n-person IM-solutions (cf. section 29). Recall Theorem 29.14 which tells us that $\{\pi^\lambda : \lambda \in \Lambda^n\}$ is the family of all solutions defined on the class G^n which satisfy PO, STI, and IM. Note that, if $\Gamma \in BSC^n$ with $S_\Gamma \in G^n$, $i \in N$, $k^i \in C^i(\Gamma)$, then also $S_{K^i(\Gamma)} \in G^n$. We have :

Theorem 30.14. Every solution $\pi^\lambda : G^n \to \mathbb{R}^n$ $(\lambda \in \Lambda^n)$ is risk sensitive on $\{\Gamma \in BSC^n : S_\Gamma \in G^n\}$.

Proof. Analogous to the proof of Theorem 24.3. ■

So far our discussion of risk properties of bargaining solutions on BSC^n. Just as in the 2-person case, less nice results can be obtained if we extend our attention to the general case of bargaining games with also risky Pareto optimal outcomes; cf. Theorem 30.4. Still, some statements can be made, comparable to a few results obtained in section 26 for 2-person bargaining solutions; we refer the reader to Peters and Tijs (1985), where also the other results of this section were published.

Finally : Nielsen (1983) shows that the 3-person, symmetric "Nash-solution" (the solution ϕ^H with $H = \langle\{1,2,3\},(\frac{1}{3},\frac{1}{3},\frac{1}{3})\rangle$ in our notation) is not risk sensitive, but does have some property like the worse alternative property; he also shows that the n-person "Kalai - Smorodinsky - solution" (the unique symmetric solution in Theorem 30.14) is risk sensitive (; all these results hold on BSC^n or $\{\Gamma \in BSC^n : S_\Gamma \in G^n\}$).

DIAGRAMS

In the only section of this chapter, we summarize the main relations between properties of bargaining solutions, as established in this monograph, with the aid of a few diagrams.

31. SOME DIAGRAMS

In the first series of diagrams, we summarize the main relations between properties of 2-person bargaining solutions defined on B, as established in sections 10 - 27. As far as risk properties are concerned, we suppose these to hold for a solution on BSC, unless stated otherwise. Before we can present these diagrams, we have to extend the results of section 25 (which were stated and proved for B_+) to the class B. We start with some extensions of old definitions, and the definition of a new property.

Definition 31.1. We call a solution $\phi : B \to \mathbb{R}^2$ *twist sensitive* (TS) if its restriction to B_+ is twist sensitive (cf. Def. 25.1). We say that ϕ has the *slice property* (SL) if its restriction to B_+ has the slice property (cf. Def. 25.4).

Definition 31.2. We call a solution $\phi : B \to \mathbb{R}^2$ *independent of non-individually rational outcomes* if $\phi(S) = \phi(S_+)$ for every $S \in B$.
(Independence of non-individually rational outcomes : INIR)

The main results of section 25, can now be modified to hold for solutions on B, as follows.

Theorem 31.3. Let $\phi : B \to \mathbb{R}^2$ satisfy PO and STI. Then, for ϕ, we have : (i) TS implies SL. (ii) TS and INIR together imply RS. (iii) RS and SL together imply TS. (iv) IIA implies TS. (v) IM implies TS.
Proof. (i) By Theorem 25.5. (ii) By Theorem 25.3. (iii) By Theorem 25.6. (iv) By Theorem 25.11. (v) By Theorem 25.12. ■

Remark 31.4. A word of caution in reading the now following diagrams is in order. Whenever a combination of properties implies a solution ϕ to belong to some family of solutions, it is supposed that solutions in this family have those properties; but, of course, they do not necessarily satisfy also stronger properties. For instance, in Fig. 31.3, PCO, PO, RA, STI, and IR, imply $\phi \in N$: but if $\phi \in N$, then it does not satisfy SA. In other words, if a ϕ would exist with the properties PCO, PO, SA, STI, and IR, it would be in N : but solutions in N are not super-additive.

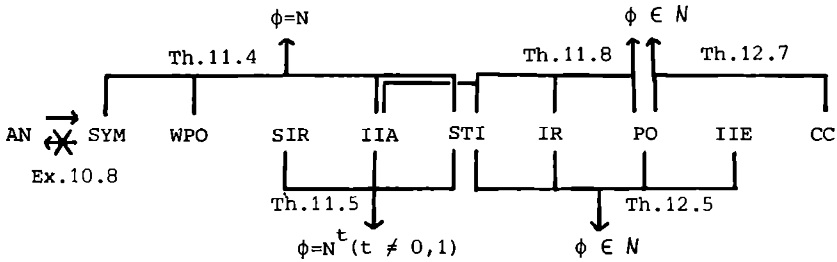


Figure 31.1 : Main results of sections 10-12, for $\phi : B \to \mathbb{R}^2$.

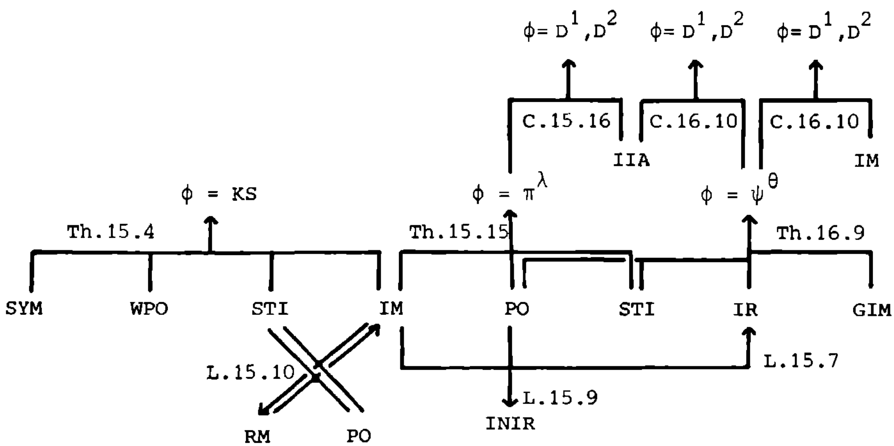
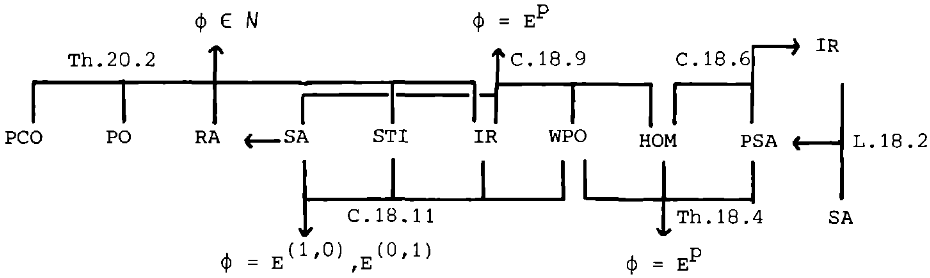
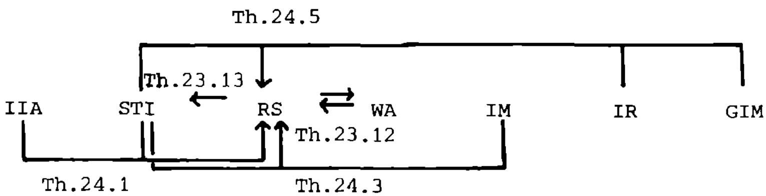


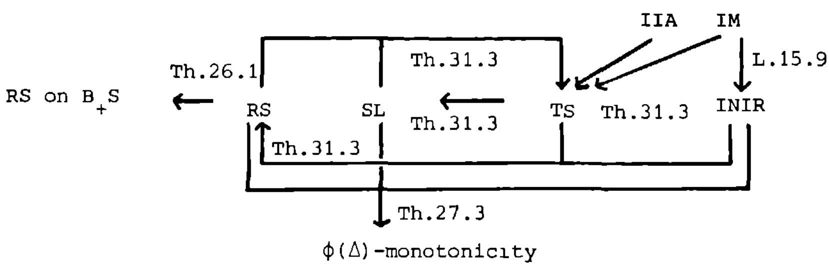
Figure 31.2. : Main results of sections 15-16, for $\phi : B \to \mathbb{R}^2$.



<u>Figure 31.3</u> : Main results of section 17-20, for $\phi : B \to \mathbb{R}^2$.



<u>Figure 31.4</u> : Main results of sections 23-24, for a PO-solution on B.



<u>Figure 31.5</u> : Main results of sections 25-27, for a PO-STI-solution on B.

In this monograph, most attention was paid to 2-person bargaining solutions. For n-person bargaining solutions, the main results are Theorems 28.23 and 29.14 which give characterizations of IIA- and IM-solutions, respectively, but on different classes of bargaining games. Further, Theorems 30.13 and 30.14 are on risk sensitivity of such solutions. Other results of section 30 are summarized in Fig. 31.6. There, whenever risk properties are concerned, these are supposed to hold on BSC^n.

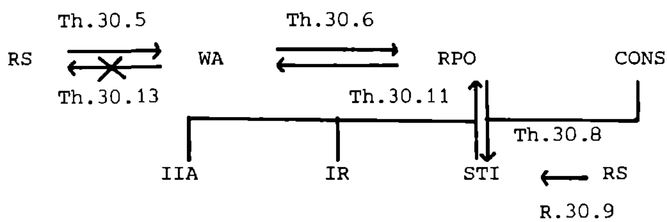


Figure 31.6 : Main results of section 30, for a PO-solution on $B^n (n > 2)$.

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AUTHOR INDEX

Dit proefschrift bestudeert verbanden tussen eigenschappen van oplossingen voor het onderhandelingsprobleem, zoals dat door Nash in 1950 geformuleerd werd. Onder meer gebeurt dit door middel van karakteriseringen van onderhandelingsoplossingen door hun eigenschappen. Onderhandelingstheorie is een onderdeel van de speltheorie, in het bijzonder van de cooperatieve theorie der spelen zonder zijdelingse betalingen.

Hoofdstuk 1 is inleidend. Hoofdstuk 2 behandelt enige ingrediënten der nutstheorie welke in dit proefschrift gebruikt worden : de von Neumann - Morgenstern nutstheorie in § 4, risico-afkerigheid in § 5, additieve en multiplicatieve nutsfuncties in respectievelijk § 6 en § 7. § 4 vormt een basis voor het hele proefschrift, § 5 voor hoofdstuk 8 en voor § 30, waarin het gaat om risico-eigenschappen van onderhandelingsoplossingen. De resultaten van § 6 en § 7 worden gebruikt in hoofdstuk 6 (over additiviteitseigenschappen) en § 12 (over karakteriseringen van zogenaamde Nash oplossingen).

In hoofdstuk 3 worden de centrale concepten van dit proefschrift geïntroduceerd : onderhandelingssituaties in § 8, onderhandelingsspelen in § 9, en onderhandelingsoplossingen in § 10.

Hoofdstuk 4 houdt zich vrijwel uitsluitend bezig met de (2-persoons) onderhandelingsoplossing van Nash en de niet-symmetrische uitbreidingen daarvan. In § 11 staat de zgn. onafhankelijkheid van irrelevante alternatieven (Nash) centraal, in § 12 spelen de eigenschappen genaamd onafhankelijkheid van irrelevante uitbreidingen (Thomson) en conventie-consistentie (Binmore) de belangrijkste rol. Alle genoemde eigenschappen van onderhandelingsoplossingen worden in karakteriseringen gebruikt. In § 13 worden verbanden gelegd tussen het niet-cooperatieve oplossingsconcept competitief evenwicht voor een eenvoudig markt-model, en onderhandelingsoplossingen, in het bijzonder Nash oplossingen. § 14 behandelt een tweede niet-cooperatief model, waarbij niet-symmetrische Nash oplossingen een belangrijke rol spelen. In de beide laatstgenoemde paragrafen worden interpretaties gegeven van de parameter van een niet-symmetrische Nash oplossing.

Hoofdstuk 5 geeft een karakterisering van een familie van niet-symmetrische uitbreidingen van de (2-persoons) Kalai-Smorodinsky oplossing (in § 15) en van de (2-persoons) Kalai-Rosenthal oplossing (in § 16); de belangrijkste daarbij gebruikte eigenschappen zijn respectievelijk individuele monotonie en globale individuele monotonie.

Hoofdstuk 6 houdt zich bezig met additiviteitseigenschappen van 2-persoons onderhandelingsoplossingen. In § 17 wordt er een nutstheoretische onderbouwing gegeven voor het gebruik van dergelijke eigenschappen, waarbij het resultaat van § 6 toegepast wordt. § 18 geeft een karakterisering van de familie van zgn. proportionele oplossingen (Kalai), m.b.v. de (partiele) additiviteitseigenschap. In § 20 wordt de familie van niet-symmetrische Nash oplossingen gekarakteriseerd met behulp van de zgn. beperkte-additiviteitseigenschap. § 19 is een intermezzo over (Pareto-) continuïteit van onderhandelingsoplossingen.

Hoofdstuk 7 bekijkt twee andere benaderingswijzen van het onderhandelingsprobleem : multi-oplossingen en probabilistische oplossingen. Een multi-oplossing voegt aan ieder onderhandelingsspel een deelverzameling van mogelijke uitkomsten toe. Een probabilistische oplossing voegt aan ieder onderhandelingsspel een kansverdeling op de verzameling van mogelijke uitkomsten toe. In § 21 worden 2-persoons multi-oplossingen bekeken welke voldoen aan een versie van de onafhankelijkheid van irrelevante alternatieven; zodoende wordt er een uitbreiding verkregen van het belangrijkste resultaat van § 11. Tevens worden er multi-oplossingen bestudeerd met de beperkte-monotonie eigenschap en wordt er een uitbreiding verkregen van de karakterisering van § 15. In § 22 worden twee versies van onafhankelijkheid van irrelevante alternatieven voor probabilistische oplossingen besproken; ook nu weer wordt de karakterisering van § 11 uitgebreid.

Hoofdstuk 8 bestudeert risico-eigenschappen en verbanden tussen risico-eigenschappen en andere eigenschappen van 2-persoons onderhandelingsoplossingen. Het fundament voor de definitie van deze risico-eigenschappen is gelegd in § 5. In § 23 worden hoofdzakelijk de risicogevoeligheidseigenschap en de slechter-alternatief eigenschap besproken, en in § 24 de verbanden tussen deze eigenschappen en andere eigenschappen, zoals onafhankelijkheid van irrelevante alternatieven en individuele monotonie. In § 25 wordt er een verband gelegd tussen risicogevoeligheid, draaigevoeligheid, en de zgn. schaafeigenschap. De meeste resultaten, tot aan § 26, gelden voor spelen waarin elke Pareto optimale uitkomst risicoloos is : in § 26 wordt het geval bekeken waarin een Pareto optimale uitkomst soms slechts te verwezenlijken is door een loterij. Tot § 27 luidt de belangrijkste vraag die in hoofdstuk 8 gesteld wordt, als volgt : welke invloed heeft het vervangen van een speler in een onderhandelingsspel door een meer risico-afkerige speler, op de oplossings-uitkomst ? In § 27 wordt, heel summier, de strategische kwestie bekeken of het voor een speler voordelig kan zijn zich meer of minder risico-afkerig voor te doen, voor zover de regels van het spel een dergelijk gedrag toelaten. De zgn. b-monotonie eigenschap, geïntroduceerd in § 13, speelt daarbij een belangrijke rol.

In hoofdstuk 9 worden enkele resultaten uit de hoofdstukken 4, 5, en 8, uitgebreid naar n-persoons onderhandelingsspelen en -oplossingen. In § 28 worden oplossingen bestudeerd welke onafhankelijk van irrelevante alternatieven en consistent zijn, in § 29 worden individueel monotone n-persoons oplossingen op een deelklasse der n-persoons onderhandelingsspelen bekeken, en in § 30 worden risico-eigenschappen van n-persoons oplossingen bestudeerd.

Hoofdstuk 10 tenslotte geeft (in § 31) een overzicht van de belangrijkste in dit proefschrift aangetoonde verbanden tussen eigenschappen van onderhandelingsoplossingen, met behulp van enkele pijldiagrammen.

CURRICULUM VITAE

De auteur van dit proefschrift werd op 31 juli 1953 geboren te Maastricht. Hij bezocht het Henric van Veldekecollege te Maastricht, waar hij in 1971 het diploma Gymnasium B behaalde. Vervolgens studeerde hij enige jaren in Amsterdam (economie en filosofie), en volbracht daarna zijn militaire dienstplicht. In 1974 begon hij met de studie wiskunde aan de Katholieke Universiteit van Nijmegen. In maart 1977 behaalde hij het diploma MO-A wiskunde, en wijdde daarna enige tijd aan de studie filosofie. In 1980 behaalde hij het kandidaatsexamen wiskunde met groot bijvak filosofie, en in maart 1982 het doctoraalexamen wiskunde, met hoofdvak mathematische besliskunde en speltheorie bij Prof. Dr. S.H. Tijs, en bijvak onder andere wetenschapsfilosofie. Zijn doctoraalscriptie was getiteld "Risicogevoeligheid van onderhandelingsoplossingen", en vormde een eerste aanzet tot dit proefschrift.

Van augustus 1982 tot januari 1984 was hij werkzaam als wetenschappelijk medewerker aan het Mathematisch Instituut van de Katholieke Universiteit van Nijmegen, waar hij onder leiding van Prof. Dr. S.H. Tijs onderzoek verrichtte op het gebied van de onderhandelingstheorie.

Vanaf januari 1984 is hij als universitair docent werkzaam aan de Economische Faculteit van de Rijksuniversiteit Limburg te Maastricht, als lid van de capaciteitsgroep Kwantitatieve Economie.

STELLINGEN

bij het proefschrift "Bargaining Game Theory" van H.J.M. Peters

I

Stelling 8.6 in [1], over uniciteit van evenwichtsprijzen in het daar geformuleerde lineaire ruilmodel, is onjuist. Een ander prijsevenwicht voor het voorbeeld in §8.6, is: $\bar{y}_1 = (\frac{10}{13},0,0)$, $\bar{y}_2 = (0,1,0)$, $\bar{y}_3 = (\frac{3}{13},0,1)$, bij de prijzen $\bar{\pi}_1 = \frac{13}{10}$, $\bar{\pi}_2 = 1$, $\bar{\pi}_3 = \frac{7}{10}$.

[1] D. Gale: The Theory of Linear Economic Models. McGraw-Hill, New York, 1960.

II

Indien, voor het lineaire ruilmodel genoemd in stelling I, extra wordt verondersteld dat een consument slechts dat goed of die goederen koopt waarvan het nut per geldeenheid maximaal is, zijn de evenwichtsprijzen wél uniek. Wiskundig geformuleerd, wil dat zeggen dat formule (1) op blz. 288 in [1] als extra veronderstelling wordt toegevoegd. Deze formule correspondeert met de, in een andere context geformuleerde, formule (3) in [2].

[2] E. Eisenberg and D. Gale: Consensus of Subjective Probabilities: The Pari-Mutuel Method. Annals of Mathematical Statistics 30, 165-168, 1959.

III

Onder de Pareto-rand van een niet-lege compacte convexe verzameling S in R^n verstaan we de verzameling $P(S) := \{x \in S$: als $y \in S$ en $y \geq x$, dan $y = x\}$. P(S) hoeft niet gesloten te zijn (zie [3]). Er geldt: P(S) is gesloten dan en slechts dan als er voor géén punt x van de afsluiting van P(S) een punt y van S bestaat met $y \geq x$ en $y_i = x_i$ voor precies één coordinaat i.

[3] K.J. Arrow, E.W. Barankin and D. Blackwell: Admissible Points of Convex Sets. Contributions to the Theory of Games II (eds. H.W. Kuhn and A.W. Tucker), Annals of Mathematics Studies 28, 87-91, Princeton University Press, Princeton, 1953.

IV

Zij $\emptyset \neq T \subset R^n$ en zij $k : T \to R$ een begrensde functie die concaaf is, dat wil zeggen $k(\Sigma_{i=1}^m p_i t^i) \geq \Sigma_{i=1}^m p_i k(t^i)$ voor elke convexe combinatie $\Sigma_{i=1}^m p_i t^i$ van elementen van T welke zelf ook een element van T is. Dan bestaat er een concave functie $\bar{k}$: conv(T) $\to$ R zo dat $\tilde{k}(t) = k(t)$ voor elke t in T.

V

De stelling van Anscombe en Aumann ([4]) voor beslissen bij onzekerheid, en ook sommige stellingen over additief ontbindbare von Neumann-Morgenstern nutsfuncties op Cartesische producten (zie [5], of [6, Stelling 5.1]), zijn eenvoudig te verkrijgen als corollaria van Stelling V in [7].

[4] F.J. Anscombe and R.J. Aumann: A Definition of Subjective Probability. Annals of Mathematical Statistics 34, 199-205, 1963.

[5] P.C. Fishburn: Independence in Utility Theory with Whole Product Sets. Operations Research 13, 28-45, 1965.

[6] R.L. Keeney and H. Raiffa: Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley New York, 1976.

[7] J.C. Harsanyi: Cardinal Welfare, Individualistic Ethics and Interpersonal Comparison of Utility. Journal of Political Economics 63, 309-321, 1955.

VI

De conclusie van Stelling 6 in [8] is onjuist. Deze conclusie is wel juist wanneer eigenschap 4 in deze stelling vervangen wordt door eigenschap 7, zoals gedaan is in [9].

[8] A.E. Roth: Axiomatic Models of Bargaining. Springer Verlag, Heidelberg. 1979.

[9] R.E. Kihlstrom, A.E. Roth and D. Schmeidler: Risk Aversion and Solutions to Nash's Bargaining Problem. Game Theory and Mathematical Economics (O. Moeschlin and D. Pallaschke, eds.), 65-71, North-Holland, 1981.

VII

In de vele economische leerboeken die speltheoretische onderwerpen als nulsomspelen en het prisoner's dilemma behandelen, zouden ook andere onderwerpen uit de speltheorie moeten worden opgenomen.

VIII

Voor werknemers die het verrichten van onbezoldigd overwerk als vanzelfsprekend beschouwen, is loonsverlaging een meer passende benaming voor datgene wat met arbeidsduurverkorting aangeduid wordt.

IX

Het verdient aanbeveling de wiskunde op een probleemgestuurde manier te onderwijzen.

X

Dank zij het moderne Nederlandse liberalisme kan het woord "partijgenoten" nu ook als voltooid deelwoord gebruikt worden.

[10] Vrijende VVD'ers in de Tweede Kamer. Playboy, april 1986.